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Propagation and interaction of waves in a relaxing gas

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Propagation of disturbances through a uniform region of a relaxing gas in a duct with spatially varying cross section is analysed using the methods of relatively undistorted waves and weakly nonlinear geometrical optics. Particular attention is focused on situations when the disturbance amplitude is finite, arbitrarily small and not so small. In certain situations a complete history of the evolutionary behaviour of waves including weak shocks can be traced out. The asymptotic decay laws for weak shocks in a non-relaxing gas are exactly recovered. The damping effects of relaxation and non-planar wavefront configurations on the distortion, attenuation and shock formation of pulses, as they propagate, are described in detail. In the small-amplitude high-frequency limit, a solution up to the second order is obtained and numerical computations are carried out for typical values of the physical parameters involved in the solution. Transport equations are derived for signals having all possible wave modes which are mutually coupled and interact resonantly among themselves. The progressive wave approach describes the far field behaviour which is governed by the generalized Burger's equation.

1. Introduction

The effect of nonlinearity on wave propagation leading to a multivalued solution has been the subject of great interest from both mathematical and physical view points. Asymptotic analysis of nonlinear hyperbolic waves has received great attention during the past decade; it has produced several new and interesting results, which may find enumerous applications in the field of continuum mechanics.

Varley & Cumberbatch (1966) introduced the theory of relatively undistorted waves to take into account nonlinear phenomena which are governed by nonlinear equations. Based on this method, which places no restriction on the wave amplitude, Dunwoody (1968) has discussed high-frequency plane waves in ideal gases with internal dissipation. This method, which has been discussed in detail by Seymour & Mortell (1975), proposes an expansion scheme which generalizes that used in linear geometrical acoustics to account for amplitude dispersion and shock formation. Seymour & Mortell (1975) have shown that by representing high-frequency waves in terms of modulated simple waves with slowly changing Riemann invariants, the parameter expansion technique of geometrical optics can be modified to include finite-amplitude waves. The modulated simple wave theory has been used by Varley & Cumberbatch (1970), Parker (1972), Parker & Seymour (1980), McCarthy (1984) and Gupta *et al.* (1992) to discuss high-frequency waves in diverse material media. Choquet-Bruhat (1969), while dealing with small-amplitude waves, proposed

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a method similar to that of Varley & Cumberbatch (1966) to discuss shockless solutions of hyperbolic systems which depend upon a single phase function. A general discussion of single phase progressive waves has been given by Germain (1971), Fusco (1982) and Hunter & Keller (1983). The progressive wave approach has been used by Sharma *et al.* (1987, 1989) for analysing the decay of a saw-tooth profile in MHD, and to describe the signal transmission in a traffic flow respectively. In this context, the pioneering work due to Ockendon & Spence (1969) and Blythe (1969), which deals with one-dimensional planar unsteady motion in a gas with internal dissipation is worth mentioning. An interesting study on the evolution of plane compressive pulses in reacting gases has been carried out by Clarke; for this, the reader is referred to his illuminating papers (Clarke 1978, 1979) and the survey article (Clarke 1984). The non-resonant multiwave theory of weakly nonlinear geometrical optics, proposed by Hunter & Keller (1983), has been further extended by Majda & Rosales (1984) and Hunter *et al.* (1986) which enables one to take into consideration systems where many waves coexist and interact with one another resonantly.

In this paper, using the related procedures, we present a detailed analysis of non-linear waves advancing into a uniform region of a relaxing gas in a duct with spatially varying cross section. The particular non-equilibrium phenomenon of interest is the vibrational relaxation of a pure inviscid gas; the rotational and translational modes are assumed to be in local thermodynamical equilibrium throughout. An attempt is made to relate and unify various approaches, which appear formally to be quite disjoint, by drawing the connection between the results obtained by using them. After dispensing with necessary preliminaries in the next section, we consider situations in which only one component wave is excited and, then, study the effects of relaxation and non-planar geometry on wave propagation. In § 3, we discuss finite-amplitude disturbances using relatively undistorted wave approximation, and derive a condition for the validity of this approximation in the small-amplitude limit. Next, we consider the implications of this theory by considering the amplitude limit to be not so small, and therefore extend the analysis of the preceding subsection to the next order. Finally, in this section, the damping effects of relaxation and wavefront curvature on the small-amplitude high-frequency pulses are examined through a formal asymptotic analysis. In § 4, we study the situation when more than one component wave is excited and consider the coupling between the interaction and distortion of the component waves that make up the motion. Attention is drawn to the connection between the results obtained here for the off resonance case, and the corresponding results obtained in § 3. In § 5, we use the ideas of § 3 to discuss low-frequency wave process. The last section consists of some final remarks and conclusions.

2. Preliminaries

We consider disturbances in a one-dimensional unsteady flow of a relaxing gas in a duct of cross sectional area $A(x)$, where x is the distance along the duct. The gas molecules have only one lagging internal mode (i.e. vibrational relaxation) and the various transport effects are negligible. The governing equations, using summation convention, can be written in the form (see Clarke 1976)

$$\partial_t V^i + \mathcal{A}_{ij}\partial_x V^j + \mathcal{B}_i = 0, \quad i, j = 1, 2, 3, 4, \tag{1}$$

where V^i and $\mathcal{B}_i$ are the components of column vectors V and $\mathcal{B}$ defined as $V = (\rho, u, \sigma, p)^{\mathrm{T}}, \mathcal{B} = (\rho u \Omega, 0, -Q, \rho a^2 u\Omega + (\gamma-1)\rho Q)^{\mathrm{T}}$ with $\Omega = A'/A$; the superscript

'T' denotes transposition. $\mathcal{A}_{ij}$ are the components of 4×4 matrix $\mathcal{A}$ with non-zero components $\mathcal{A}_{11} = \mathcal{A}_{22} = \mathcal{A}_{33} = \mathcal{A}_{44} = u$, $\mathcal{A}_{12} = \rho$, $\mathcal{A}_{24} = 1/\rho$ and $\mathcal{A}_{42} = \rho a^2$. Here u is the particle velocity along x-axis, t the time, ρ the density, p the pressure, σ the vibrational energy, γ the frozen specific heat ratio and $a = (\gamma p/\rho)^{1/2}$ the frozen speed of sound in the gas. A prime denotes an ordinary derivative with respect to x, and the operator ∂ with a letter subscript denotes partial differentiation with respect to the indicated variable. The quantity Q, which is a known function of p, ρ and σ, denotes the rate of change of vibrational energy. The situation $Q = 0$ corresponds to a physical process involving no relaxation; indeed, it includes both the cases in which the vibrational mode is either inactive or follows the translational mode according as the flow is either frozen (σ = const.) or in equilibrium ($\sigma = \sigma^*$), where σ^* is the equilibrium value of σ evaluated at local p and ρ. Following Johannesen & Scott (1978) and Scott & Johannesen (1982), the entity Q is given by

$$Q = (\sigma^* - \sigma)/\tau, \quad \sigma^* = \sigma_0 + c(\rho\rho_0)^{-1}(p\rho_0 - \rho p_0), \tag{2}$$

where the suffix '0' refers to the initial rest condition, and the quantities τ and c, which are respectively the relaxation time and the ratio of vibrational specific heat to the specific gas constant, are assumed to be constant.

3. Relatively undistorted waves

The method of relatively undistorted waves was introduced by Varley & Cumberbatch (1966) to take into account the amplitude dispersion and possible shock formation. The advantage of this method, which makes no assumption on the magnitude of a disturbance, lies in the fact that the solution can be obtained by solving ordinary differential equations. Indeed, the solution vector V is said to define a relatively undistorted wave, if there exists a family of propagating wavelets $\phi(x,t)$ = const., such that the magnitude of the rate of change of V moving with the wavelet is small compared with the magnitude of rate of change of V at fixed t. Let us consider the transformation, $x = x$, $t = T(x,\phi)$, from (x,t) to (x,ϕ) coordinate system, and let $V(x,t) = \overline{V}(x,\phi)$. Then a relatively undistorted wave is defined by the relation $|\partial_x \overline{V}^i| \ll |\partial_x V^i|$. Thus, in a relatively undistorted wave $\partial_x V \simeq (\partial_x T)(\partial_t V)$, and consequently

$$|\partial_x \overline{V}^i| \ll |\partial_t V^i|. \tag{3}$$

In terms of independent variables (x,ϕ), equation (1) becomes

$$(\mathcal{A}_{ij}(\partial_x T) - \delta_{ij})\partial_t V^j = \mathcal{A}_{ij}\partial_x \overline{V}^j + \mathcal{B}_i, \tag{4}$$

where δ_{ij} is the kronecker function. Equations (3) and (4) are compatible if $|\mathcal{B}_i| = O(|\mathcal{A}_{ij}\partial_x \overline{V}^j|)$, while $(\partial_x T)^{-1}$ is an eigenvalue of $\mathcal{A}$. Otherwise, equation (4) would completely determine the $\partial_t V^i$ as linear forms in the $\partial_x \overline{V}^i$, and (3) could not hold. Accordingly, in such waves, where $\det|\mathcal{A}_{\rm ij} - \delta_{\rm ij}(\partial_{\rm x}\mathrm{T})^{-1}| = 0$, $\overline{V}^i$ must satisfy the compatibility condition

$$L_i(\mathcal{A}_{ij}\partial_x \overline{V}^j + \mathcal{B}_i) = 0, \tag{5}$$

for every left eigenvector L of $\mathcal{A}$ corresponding to the eigenvalue $(\partial_x T)^{-1}$. Thus the essential idea underlying in this method is based on a scheme of successive approximations to the system (4) which, to a first approximation, is replaced by

$$(\mathcal{A}_{ij} - \delta_{ij}(\partial_x T)^{-1})\partial_\phi \overline{V}^j = 0. \tag{6}$$

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Then, if $R = (R_i)$ is the right eigenvector of $\mathcal{A}$ corresponding to the eigenvalue $(\partial_x T)^{-1}$, equation (6) implies that to a first approximation

$$\partial_\phi \overline{V}^{\,i} = k(\phi, x) R_i, \tag{7}$$

for some scalar $k(\phi, x)$. It is important to appreciate that equations (7) are approximate and can not, in general, be integrated to obtain relations in $\overline{V}^{\,i}$ which are uniformly valid for all time. The terms which have been neglected in arriving at (7) will, in general, ultimately produce first-order contributions to $\overline{V}^{\,i}$. However, these contributions are negligible whenever the wave is a pulse or a high-frequency disturbance (see Varley & Cumberbatch 1970).

The matrix $\mathcal{A}$ has eigenvalues $u \pm a$ and u (with multiplicity two); here we are concerned with the solution in the region $x > x_0$, where a motion consisting of only one component wave, associated with the eigenvalue $(\partial_x T)^{-1} = u + a$, is perturbed at the boundary $x = x_0$ by an applied pressure $p(x_0, t) = \Pi(t)$. It may be noted that for nonlinear systems, there is, in general, no superposition principle so that when more than one wave mode is excited, the propagation of the individual component waves can not be calculated independently. Consequently, the problem involving nonlinear interaction of component waves needs a different approach, which is outlined in § 5. The left and right eigenvectors of A corresponding to the eigenvalue

$$(\partial_x T)^{-1} = u + a \tag{8}$$

are

$$L = (0, \rho a, 0, 1), \quad R = (\rho, a, 0, \rho a^2)^{\mathrm{T}}. \tag{9}$$

(a) *Finite-amplitude disturbances*

Let us consider the situation when the disturbance, which is headed by the front $\phi(x, t) = 0$, is moving into a region, where before its arrival the gas is in a uniform state at rest with $u = 0$, $p = p_0$, $\rho = \rho_0$ and $\sigma = \sigma_0$. It is possible to choose the label, ϕ, of each wavelet so that $\phi = t$ on $x = x_0$; consequently the boundary conditions for p and T become

$$p = \Pi(\phi), \quad T = \phi \quad at \quad x = x_0. \tag{10}$$

To a first approximation, conditions in the wave region, associated with $(\partial_x T)^{-1} = u + a$, are determined by the differential relations (7), which can be formally integrated subject to the uniform reference values $u = 0$, $p = p_0$, $\rho = \rho_0$ and $\sigma = \sigma_0$ on the leading front $\phi = 0$. Thus we have

$$\left.\begin{aligned} \rho = \rho_0 (p/p_0)^{1/\gamma}, \quad u = 2a_0(\gamma - 1)^{-1}\{(p/p_0)^{(\gamma-1)/2\gamma} - 1\}, \\ a = a_0 (p/p_0)^{(\gamma-1)/2\gamma}, \quad \sigma = \sigma_0, \end{aligned}\right\} \tag{11}$$

which hold at any x on any wavelet $\phi =$ const. Equation (5), in view of $(9)_1$ and (11), provides the following transport equations for the variation of p and T at each wavelet $\phi =$ const.:

$$\partial_x p + \frac{(\gamma - 1)\rho_0}{4a_0 F(p)} \left(\frac{p}{p_0}\right)^{1/\gamma} \times \left\{ \frac{2a_0^3 \Omega}{\gamma - 1} \left(\frac{p}{p_0}\right)^{(\gamma-1)/\gamma} \left[\left(\frac{p}{p_0}\right)^{(\gamma-1)/2\gamma} - 1 \right] + (\gamma - 1) Q \right\} = 0, \tag{12}$$

$$\partial_x T = (\gamma - 1)/2a_0 F(p), \tag{13}$$

where $F(p) = \{((\gamma+1)/2)(p/p_0)^{(\gamma-1)/2\gamma} - 1\}$. Equation (12), on using (2) and (11), may be integrated by using the boundary condition $(10)_1$ to give $p = P(x, \Pi(\phi))$; once p is known, equation (13) may be integrated subject to $(10)_2$ to determine $t = T(x, \phi)$. Subsequently $p(x,t)$ may be obtained, and hence $\rho(x,t)$, $u(x,t)$ and $a(x,t)$. As a matter of fact, the integration of (13) leads to the determination of the location of ϕ wavelets in the form

$$T = \phi + \int_{x_0}^{x} \Lambda(s, \Pi(\phi))\,\mathrm{d}s,$$

where $\Lambda = (\gamma - 1)/(2a_0 F(P(s, \Pi(\phi))))$.

From this result, it follows immediately that a shock forms on the wavelet ϕ_s at the point x_s, where

$$1 + \Pi' \int_{x_0}^{x_s} \partial_\Pi \Lambda(X, \Pi(\phi_s))\,\mathrm{d}X = 0.$$

The above results indicate that both the amplitude dispersion and shock formation along any wavelet depend on the amplitude, $\Pi(\phi)$, carried by that wavelet. It is discussed in detail in a subsection below, after we have first discussed the small-amplitude simplifications.

(*b*) *Small-amplitude waves*

In the small-amplitude limit, equations (12) and (13) can be linearized about the uniform reference state $p = p_0$, $\rho = \rho_0$, $\sigma = \sigma_0$, $u = 0$, to yield

$$\partial_x p_1 + (\alpha + \tfrac{1}{2}\Omega)p_1 = 0, \quad (\partial_x T) = \left\{1 - (\gamma+1)(2\rho_0 a_0^2)^{-1} p_1\right\} a_0^{-1}, \tag{14}$$

where p_1 is the small perturbation of the equlibrium value p_0 and $\alpha = (\gamma-1)^2 c/2\gamma a_0 \tau$, which serves as the amplitude attenuation rate on account of relaxation; it may be noted that α is identical with the absorption rate defined by Johannesen & Scott (1978) and Scott & Johannesen (1982). The ideal gas case corresponds to $\alpha = 0$. The boundary conditions for p_1 and T which follow from (10) can be rewritten as

$$p_1 = \hat{\Pi}(\phi), \quad T = \phi \quad \text{at} \quad x = x_0, \tag{15}$$

where $|\hat{\Pi}(\phi)| = |\Pi(\phi) - p_0| \ll 1$. Equations (14), together with (15), yield on integration

$$p_1 = \hat{\Pi}(\phi)\psi(x), \tag{16}$$

$$a_0(T - \phi) = x - x_0 - (\gamma+1)(\hat{\Pi}(\phi)/2\rho_0 a_0^2) J(x_0, x, \alpha), \tag{17}$$

where

$$J(x_0, x, \alpha) = \int_{x_0}^{x} (A(s)/A_0)^{-1/2} \exp(-\alpha(s - x_0))\,\mathrm{d}s$$

and

$$\psi(x) = (A/A_0)^{-1/2} \exp(-\alpha(x - x_0)).$$

Equation (16) implies that for each wavelet $\phi = \text{const.}$, which decays exponentially, the attenuation factor is independent of the amplitude $\hat{\Pi}(\phi)$ carried by the wavelet. However, conditions at any x on a wavelet $\phi_1 = \text{const.}$ are determined by the signal carried by ϕ_1, and are independent of the precursor wavelets $0 \leqslant \phi < \phi_1$.

It may be recalled that for plane and radially symmetric flow configurations,

$(A/A_0) = (x/x_0)^m$; and therefore the integral J converges to a finite limit J_0 as $x \to \infty$, i.e.

$$\lim_{x\to\infty} J(x_0, x, \alpha) \equiv J_0 = \begin{cases} 1/\alpha, & \text{(plane)};\\ (\pi x_0/\alpha)^{1/2}\exp(\alpha x_0)\mathrm{erfc}\,(\sqrt{\alpha x_0}), & \text{(cylindrical)};\\ x_0 E(\alpha x_0)\exp(\alpha x_0), & \text{(spherical)}; \end{cases} \tag{18}$$

where

$$\mathrm{erfc}\,(\mathrm{x}) = 2\pi^{-1/2}\int_{\mathrm{x}}^{\infty}\exp(-\mathrm{t}^2)\,\mathrm{dt} \quad \text{and} \quad \mathrm{E(x)} = \int_{\mathrm{x}}^{\infty}\mathrm{t}^{-1}\exp(-\mathrm{t})\,\mathrm{dt}$$

are, respectively, the complementary error function and the exponential integral. Thus J can be expressed as

$$J = J_0(1 - K(x)), \tag{19}$$

where

$$K(x) = \begin{cases} \exp(-\alpha(x - x_0)), & \text{(plane)};\\ \mathrm{erfc}\,(\sqrt{\alpha x})/\mathrm{erfc}\,(\sqrt{\alpha x_0}), & \text{(cylindrical)};\\ E(\alpha x)/E(\alpha x_0), & \text{(spherical)}. \end{cases}$$

Equation (17) indicates that a shock first forms at (x_s, ϕ), where the minimum value of x (i.e. x_s) is given by the solution of

$$H(x) \equiv 1 - (\hat{\Pi}'(\phi)/b)(1 - K(x)) = 0, \tag{20}$$

where $b = 2\rho_0 a_0^3/(\gamma+1)J_0 > 0$ and $\hat{\Pi}'(\phi) \equiv \mathrm{d}\hat{\Pi}/\mathrm{d}\phi$; since the expression $(1 - K(x))$ monotonically increases from 0 to 1 as x increases from x_0 to ∞, it follows that a shock can only occur if $\hat{\Pi}' > b > 0$, and subsequently $H(x)$ first vanishes at that wavelet ϕ where $\hat{\Pi}'(\phi)$ is greatest. Expression for the shock formation distance on the leading wavefront $\phi = 0$, where (20) is exact, was pointed out by Johannesen & Scott (1978). It may be noted that the relatively undistorted approximation is valid only if

$$|\hat{\Pi}'/\hat{\Pi}| \gg |(\alpha + \tfrac{1}{2}\Omega)\{1 + (\gamma + 1)(2\rho_0 a_0^2)^{-1}\hat{\Pi}\psi(x)\}a_0 H|, \tag{21}$$

which, in fact, corresponds to the slow modulation approximation (Varley & Cumberbatch 1970). As discussed above, a shock wave may be initiated in the flow region, and once it is formed, it will propagate by separating the portions of the continuous region. When the shock is weak, its location can be found from the equal area rule (see Whitham 1974):

$$2\int_{\phi_1}^{\phi_2}\hat{\Pi}(t)\,\mathrm{d}t = (\phi_2 - \phi_1)\{\hat{\Pi}(\phi_1) + \hat{\Pi}(\phi_2)\}, \tag{22}$$

where ϕ_1 and ϕ_2 are the wavelets ahead of and behind the shock. For a weak shock propagating into an undisturbed region, where $\hat{\Pi}(\phi_1) = 0$ for $\phi_1 \leqslant 0$, equation (22), on using (17) and (19), becomes

$$\int_0^{\phi_2}\hat{\Pi}(t)\,\mathrm{d}t = (\gamma + 1)\hat{\Pi}^2(\phi_2)(1 - K(x))J_0/4\rho_0 a_0^3. \tag{23}$$

This equation, in the limit $\alpha \to 0$, yields a result which agrees fully with the result obtained by Whitham (1974) for non-relaxing gases. Assuming that the integral on

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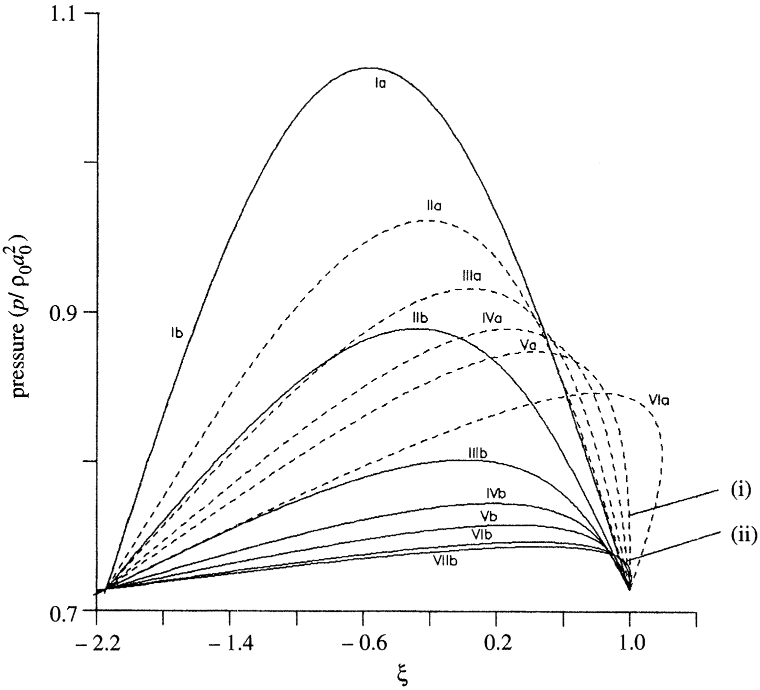


Figure 1. The variation of the dimensionless pressure $\hat{p}$ (defined as $(p_0+p_1)/\rho_0 a_0^2$) against the dimensionless variable ξ (defined as $(x-a_0 t)/x_0$), using the initial profile Ia (defined in (25)). The distortion of the profile is delineated at various distances before and after shock formation on the leading wavelet, $\hat{\phi}$=0, in cylindricallly ($m=1$) and spherically ($m=2$) symmetric flow configurations of a non-relaxing gas ($\hat{\alpha}=0$) for $\hat{\delta}=0.35$ and $\gamma=1.4$. Shock forms at (i) $\hat{x}_s=4.79$ (cylindrical) and (ii) $\hat{x}_s=10.81$ (spherical). For $m=1$, $\hat{x}=1$ (Ia), $\hat{x}=2$ (IIa), $\hat{x}=3$ (IIIa), $\hat{x}=4$ (IVa), $\hat{x}=4.79$ (Va), $\hat{x}=7$ (VIa); for $m=2$, $\hat{x}=1$ (Ib), $\hat{x}=2$ (IIb), $\hat{x}=4$ (IIIb), $\hat{x}=6$ (IVb), $\hat{x}=8$ (Vb), $\hat{x}=10.81$ (VIb), $\hat{x}=12$ (VIIb).

the left-hand side of (23) is bounded, it follows that for sufficiently large x, $\hat{\Pi}(\phi_2)$ stays in proportion to $J_0^{1/2}$. Consequently, in view of (13) and (16), the pressure jump, $[p]$, across the shock, which is defined as the measure of shock amplitude, decays like

$$[p] \propto \begin{cases} \exp(-\hat{\alpha}\hat{x}) & \text{(plane);} \\ \hat{x}^{-1/2}\exp(-\hat{\alpha}\hat{x}) & \text{(cylindrical);} \\ \hat{x}^{-1}\exp(-\hat{\alpha}\hat{x}) & \text{(spherical);} \end{cases} \tag{24}$$

where $\hat{\alpha}=\alpha x_0$ and $\hat{x}=x/x_0$. For a non-relaxing gas (α =0), we find that as $x\to\infty$, the shock decays like

$$[p] \sim x_0 \begin{cases} \hat{x}^{-1/2} & \text{(plane);} \\ \hat{x}^{-3/4} & \text{(cylindrical);} \\ \hat{x}^{-1}(\ln\hat{x})^{-1/2} & \text{(spherical).} \end{cases}$$

These asymptotic results for non-relaxing gases are in full accord with the earlier results (Whitham 1974).

In order to trace the early history of shock decay after its formation on the leading wave front $\phi=0$, we consider a special case in which the disturbance at the boundary

$x = x_0$ is a pulse defined as

$$\hat{\Pi}(\phi) = \begin{cases} 0, & \phi < 0, \\ \delta \sin(a_0\phi/x_0), & 0 < \phi < \pi x_0/a_0, \qquad \delta > 0. \\ 0, & \phi > \pi x_0/a_0, \end{cases} \tag{25}$$

It may be recalled that $H(x)$, in equation (20), first vanishes on that wavelet for which $\hat{\Pi}'(\phi)$ has a maximum value greater than b, i.e. $\delta a_0/x_0 > b$; consequently the shock first forms on the wavelet $\phi = 0$ at a distance $x = x_s$, nearest to x_0, given by the solution of

$$E(x) \equiv (\tfrac{1}{2}(\gamma+1))(1-K(x))\hat{\delta}\hat{J}_0 = 1, \tag{26}$$

where $\hat{\delta} = \delta/\rho_0 a_0^2$ and $\hat{J}_0 = J_0/x_0$ are the dimensionless constants. The distortion of the pulse, defined in (25), is shown in figures 1 and 2. The usual steepening of the compressive phase and flattening of the expansive phase of the wavelet are quite evident from the distorted profiles. The depression and flattening of the peaks with increasing $\hat{x}$, which become even more pronounced on account of relaxation or the wavefront geometry, indicate that the disturbance is undergoing a general attenuation. Equation (20) shows that for $\hat{\alpha} = 0$, the pressure profile develops a vertical slope, thereby indicating the appearence of a shock at $\hat{x} = 4.79$ and 10.81 in cylindrically and spherically symmetric flows respectively, while for $\hat{\alpha} = 0.05$, shocks in respective flow configurations develop at $\hat{x} = 5.31$ and 20.92. Thus, the presence of relaxation or an increase in the wavefront curvature both serve to delay the onset of a shock. Equation (23), in view of (25), then simplifies to give ϕ_2 on the shock by the following relation,

$$\sin \hat{\phi}_2 = 2(E(x)-1)^{1/2}/E(x), \tag{27}$$

where $\hat{\phi}_2 = \phi_2 a_0/x_0$ is the dimensionless variable. Equation (27), together with (16) and (25), implies

$$[\hat{p}] = 2\hat{\delta}\hat{x}^{-m/2}\exp(-\hat{\alpha}(\hat{x}-1))(E(x)-1)^{1/2}/E(x), \tag{28}$$

where $\hat{p} = p/\rho_0 a_0^2$ is the dimensionless pressure. Equation (28) shows that the shock after its formation on $\phi = 0$, at $x = x_s > x_0$, grows to a maximum strength at $x = x_1 > x_s$, where x_1 is given by the solution of $E(x) = 2$, and then decays ultimately in proportion to $x^{-m/2}\exp(-\alpha x)$ as concluded above.

Let us now consider a special case in which the small disturbance at the boundary, $x = x_0$, has a periodic wave form given by

$$\hat{\Pi}(\phi) = \overline{\delta}\sin(\hat{\phi}), \tag{29}$$

where $\overline{\delta} < 0$ and $\hat{\phi} = a_0\phi/x_0$, and consider the development over one cycle $0 \leqslant \hat{\phi} \leqslant 2\pi$. In this case, the shock first forms on the wavelet $\hat{\phi} = \pi$ at a distance $x = x_s$, nearest to x_0, given by the solution of (20); this can occur, of course, if $|\overline{\delta}|a_0/x_0 > b$. Equations (17) and (22) are satisfied on the shock, if $\hat{\phi}_1 + \hat{\phi}_2 = 2\pi$ and $\hat{\phi}_1 - \hat{\phi}_2 = 2\theta$, where θ is given by the solution of

$$\theta/\sin\theta = \tfrac{1}{2}(\gamma+1)(1-K(x))\hat{J}_0|\tilde{\delta}|, \tag{30}$$

with $\tilde{\delta} = \overline{\delta}/\rho_0 a_0^2$. The discontinuity in $\hat{p}$ at the shock is, therefore, given by

$$[\hat{p}] = 2|\tilde{\delta}|(x/x_0)^{-m/2}\exp(-\alpha(x-x_0))\sin\theta, \tag{31}$$

which shows that the shock starts with zero strength corresponding to $\theta \to 0$ at

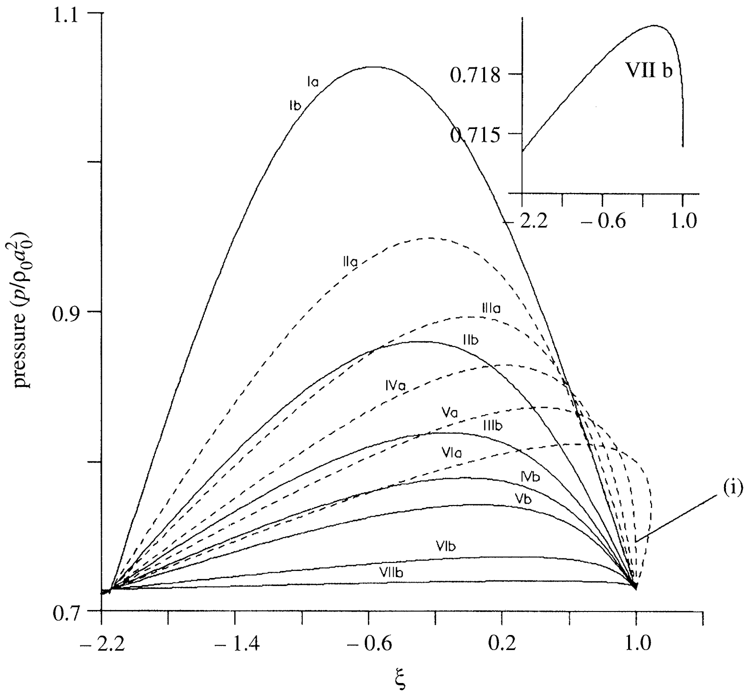


Figure 2. Development of the pressure profile $\hat{p}$ against ξ, using the same initial profile (as in figure 1), at the leading wavehead for $\tilde{\delta} = 0.35$, $\hat{\alpha} = 0.05$ and $\gamma = 1.4$: the figure in the inset shows the development on the wavehead, $\hat{\phi} = 0$, for $m = 2$ when the profile Vb develops a vertical slope signifying the appearence of a shock at $\hat{x} = 20.92$ in a relaxing gas. Shock forms at (i) $\hat{x}_s = 5.31$ (cylindrical). For $m = 1$, $\hat{x} = 1$ (Ia), $\hat{x} = 2$ (IIa), $\hat{x} = 3$ (IIIa), $\hat{x} = 4$ (IVa), $\hat{x} = 5.31$ (Va), $\hat{x} = 7$ (VIa); for $m = 2$, $\hat{x} = 1$ (Ib), $\hat{x} = 2$ (IIb), $\hat{x} = 3$ (IIIb), $\hat{x} = 4$ (IVb), $\hat{x} = 5$ (Vb), $\hat{x} = 10$ (VIb), $\hat{x} = 20.92$ (VIIb).

$x = x_s$, given by the solution of (30). The shock amplitude decays to zero as $\theta \to \theta_m$, where θ_m is given by the solution of $\theta_m/\sin(\theta_m) = \frac{1}{2}(\gamma+1)|\tilde{\delta}|\hat{J}_0$. The shock strength grows for θ lying in the interval $0 < \theta < \theta_*$, whereas it decays over the interval $\theta_* < \theta < \theta_m$, thus exhibiting a maximum corresponding to $\theta = \theta_*$ at a distance $x = x_*$, where both θ_* and x_* can be determined using (30) and the relation

$$4(\hat{\alpha} + m/2\hat{x}_*)(\sin\theta_* - \theta_*\cos\theta_*) = (\gamma+1)|\tilde{\delta}|\hat{x}_*^{-m/2}\sin(2\theta_*)\exp(-\hat{\alpha}(\hat{x}_* - 1)).$$

The shock decays ultimately with $\theta = \theta_m$, $x \to \infty$, according to the law

$$[\hat{p}] \sim \lambda\hat{x}^{-m/2}\exp(-\hat{\alpha}\hat{x}), \tag{32}$$

where $\lambda = 4\theta_m/(\gamma+1)\hat{J}_0$. However, in the absence of relaxation, it follows from (30) and (31) that as $\theta \to \pi$, $x \to \infty$, the shock decays like

$$[\hat{p}] \sim 4\pi(\gamma+1)^{-1}\begin{cases}\hat{x}^{-1} & \text{(plane);}\\ (2\hat{x})^{-1} & \text{(cylindrical);}\\ (\hat{x}\ln\hat{x})^{-1} & \text{(spherical).}\end{cases} \tag{33}$$

The development of the pressure profile with initial disturbance given by (29), and the subsequent shock formation at the wavehead $\hat{\phi} = \pi$ are exhibited in figures 3*a* (with relaxation) and 3*b* (without relaxation), showing in effect that the influence of

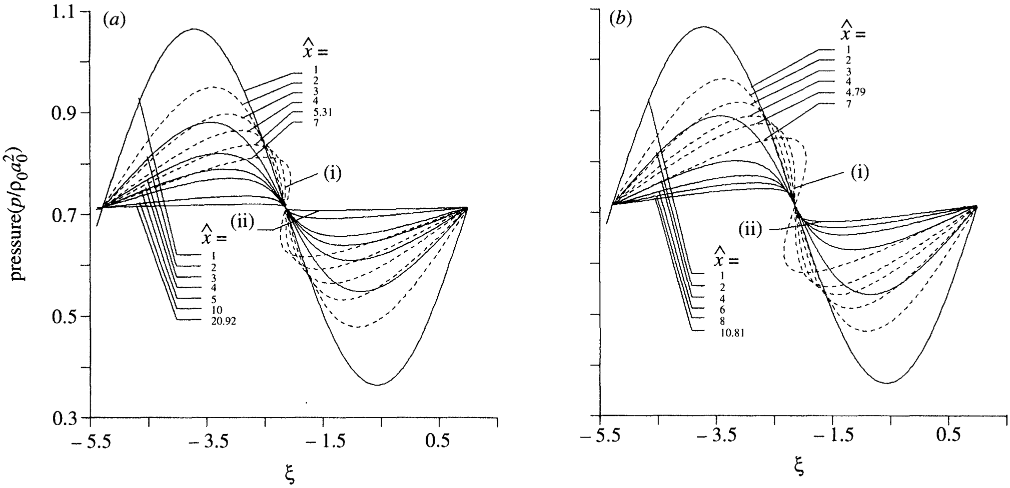


Figure 3. (a) The variation of the dimensionless pressure $\hat{p}$ against the dimensionless variable ξ using the initial profile (defined in (29)). The distortion of the profile is exhibited at various distances before and after shock formation on the wavelet, $\hat{\phi} = \pi$, in cylindrically and spherically symmetric flow in a relaxing gas for $\tilde{\delta} = -0.35$, $\hat{\alpha} = 0.05$ and $\gamma = 1.4$. Shock forms at (i) $\hat{x}_s = 5.31$ (cylindrical) and (ii) $\hat{x}_s = 20.92$ (spherical). (b) Development of the pressure profile $\hat{p}$ against ξ, using the same initial profile (as in (a)), at the wavehead, $\hat{\phi} = \pi$, for $\tilde{\delta} = -0.35, \hat{\alpha} = 0$ and $\gamma = 1.4$. Shock forms at (i) $\hat{x}_s = 4.79$ (cylindrical) and (ii) $\hat{x}_s = 10.81$ (spherical).

relaxation is to delay the onset of a shock wave. The evolutionary behaviour of the pressure profile before and after the shock formation, exhibited in figure 3a, follows a slightly different pattern from that depicted in figure 3b, in the sense that the profile, which eventually folds into itself, develops concavity with the peak slightly advanced. The evolutionary behaviour of shocks evolving from profile (29) are depicted in figure 4; indeed, the shock after its formation at $x = x_s$ grows to a maximum strength at $x = x_*$, and then decays according to the law (32) or (33) depending on whether the gas is relaxing or non-relaxing respectively. However, in the absence of relaxation, a shock resulting from (29) decays faster than that evolving from the pulse (25).

(c) *Waves with amplitude not-so-small*

We now seek to explore the predictions of § 3 a by considering the amplitude limit to be not so small; in fact, here we extend the analysis of the preceding subsection to the next order by including the nonlinear quadratic terms in the perturbed flow quantities p_1, ρ_1, etc., which were otherwise ignored in the § 3 b while writing equations (14). In this instance, equations (12) and (13) reduce to

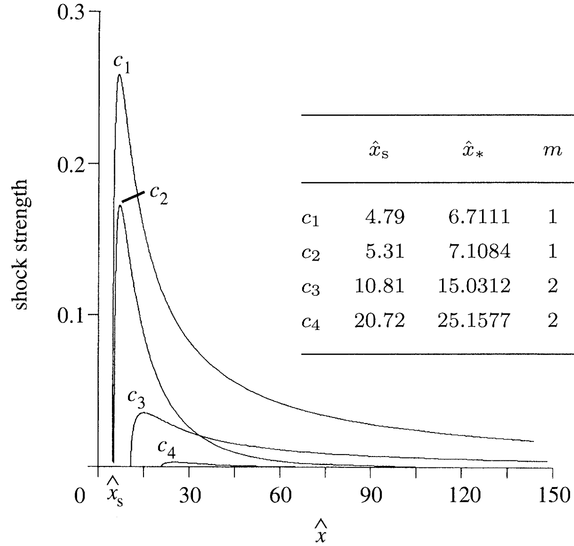
$$\partial_x p_1 + (\alpha + \tfrac{1}{2}\Omega)p_1 - q(x)p_1^2 = 0, \tag{34}$$

$$\partial_x T = \{1 - (\gamma+1)(2\rho_0 a_0^2)^{-1} p_1 + 3(\gamma+1)^2(8\rho_0^2 a_0^4)^{-1} p_1^2\}/a_0, \tag{35}$$

where $q(x) = \{\alpha\gamma + \frac{1}{4}(3-\gamma)\Omega\}/2\rho_0 a_0^2$. Equations (34) and (35), together with the boundary conditions for p_1 and T at $x = x_0$, yield on integration

$$p_1 = \hat{\Pi}(\phi)\psi(x)\{1 - \hat{\Pi}(\phi)M(x)\}^{-1}, \tag{36}$$

$$a_0(T - \phi) = (x - x_0) + \frac{\gamma+1}{2\rho_0 a_0^2}\int_{x_0}^{x}\left(\frac{3(\gamma+1)}{4\rho_0 a_0^2}p_1^2 - p_1\right)\mathrm{d}x, \tag{37}$$



	$\hat{x}_s$	$\hat{x}_*$	m	$\hat{\alpha}$
c_1	4.79	6.7111	1	0.00
c_2	5.31	7.1084	1	0.05
c_3	10.81	15.0312	2	0.00
c_4	20.72	25.1577	2	0.05

Figure 4. Growth and decay of a shock wave which appears first at $\hat{x} = \hat{x}_s$ on the wavelet $\hat{\phi} = \pi$; comparison is made with the behaviour from the same initial and boundary data for a non-relaxing ($\hat{\alpha} = 0$) gas. The effects of a relaxation and the wavefront curvature on the shock formation distance ($\hat{x}_s$), and the distance ($\hat{x}_*$) at which the shock strength attains maximum strength are exhibited; $\hat{\delta} = -0.35$.

where

$$M(x) = \int_{x_0}^{x} \psi(x')q(x')\,\mathrm{d}x'$$

and ϕ is the parameter distinguishing the wavelets from each other. Equations (36) and (37) indicate that in contrast to the results of § 3 *b*, both the rate at which the amplitude varies on any wavelet and the time taken to form a shock are influenced by the amplitude of the signal carried by the wavelet. Indeed, for $\hat{\Pi} M < 0$ (respectively, > 0) the amplitude decays more rapidly (respectively, slowly) than that predicted in § 3 *b*; the computed results are shown in figures 5*a*, *b*. The results computed for small-amplitude disturbances in § 3 *b* are also incorporated into figures 5*a*, *b* for the sake of comparison and completeness. The numerical results indicate that the shock arrival time on a particular wavelet increases as compared to the case discussed in § 3 *b*. Behind the shock the wavelets are determined by (37). According to Pfriem's rule, the shock velocity $\mathrm{d}T/\mathrm{d}x|_s$ of a weak shock is given by the average of the characteristic speeds ahead of and behind the shock. Thus, to the present approximation

$$\mathrm{d}T/\mathrm{d}x|_s = \{1 - (\gamma+1)(4\rho_0 a_0^2)^{-1}(p_1 - 3(\gamma+1)p_1^2/4\rho_0 a_0^2)\}/a_0, \tag{38}$$

where $p_1 = p_1(x, \phi_s)$ denotes the value at position x on the shock. Evaluating (37) along the shock, we obtain the shock trajectory $T = T(x)$ and, thus, an alternative expression for the shock speed

$$\left.\frac{\mathrm{d}T}{\mathrm{d}x}\right|_s = \frac{\mathrm{d}\phi_s}{\mathrm{d}x}\left\{1 - \frac{\gamma+1}{2\rho_0 a_0^3}\int_{x_0}^{x}(\partial_\phi p_1)\,\mathrm{d}x + \frac{3(\gamma+1)^2}{4\rho_0^2 a_0^5}\int_{x_0}^{x} p_1(\partial_\phi p_1)\,\mathrm{d}x\right\} + \frac{1}{a_0}\left\{1 - \frac{\gamma+1}{2\rho_0 a_0^2}p_1 + \frac{3(\gamma+1)^2}{8\rho_0^2 a_0^4}p_1^2\right\}. \tag{39}$$

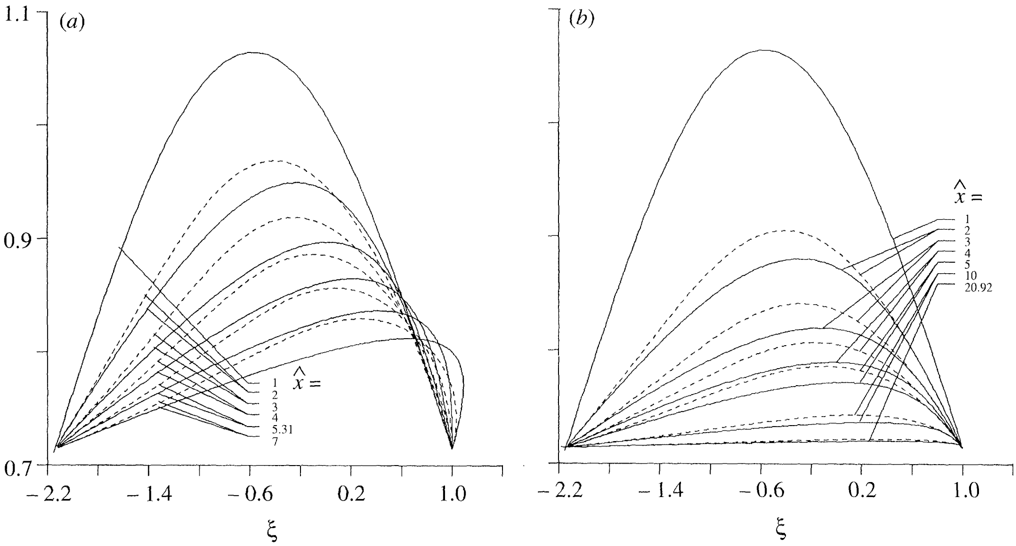


Figure 5. (a) Solution for the pressure in pulse region of a cylindrically symmetric ($m = 1$) flow of a relaxing gas at various distances when the disturbance amplitude is not-so-small (see the dashed lines). Comparision is shown with the corresponding situation when the disturbance amplitude is small (see the solid lines); the initial profile is defined in (25). (b) Solution for the pressure in pulse region of a spherically symmetric ($m = 2$) flow of a relaxing gas at various distances when the disturbance amplitude is not-so-small (see the dashed lines). Comparision is shown with the corresponding situation when the disturbance amplitude is small (see the solid lines); the initial profile is defined in (25). For both (a) and (b): $\hat{\alpha} = 0.05$, $\hat{\delta} = 0.35$, $\gamma = 1.4$.

Eliminating $\mathrm{d}T/\mathrm{d}x|_{\mathrm{s}}$ between (38) and (39), we obtain a differential equation for the unknown ϕ_{s}, which can be integrated numerically subject to the condition, $x = x_{\mathrm{s}}$ at ϕ=0. Having determined ϕ_{s} at different locations x, the shock strength at these respective locations and hence the evolutionary behaviour of shock decay can be determined from (36).

4. Nonlinear geometrical acoustic solution

Here, we use a systematic procedure to discuss a small-amplitude wavelike disturbance governed by the system (1) in the high-frequency or geometrical acoustics limit, i.e. when the time scale, τ, defined by the relaxation mechanism is large compared with the time τ_a associated with the boundary data. The geometrical acoustics limit then corresponds to the high-frequency condition $\epsilon = \tau_a/\tau \ll 1$. In this limit, the perturbations of ρ, u, p and σ caused by the wave are of size $O(\epsilon)$, and they depend significantly on a fast characteristic variable $\xi = \phi/\epsilon$. We therefore make a change of independent variables $(x,t) \to (x,\xi)$ by defining $x = x$, $t = \tilde{T}(x,\xi)$, and let $V(x,t) = \tilde{V}(x,\xi)$. It may be noted that in the high-frequency limit, V is relatively undistorted because $|\partial_x \tilde{V}|$, $|\mathcal{B}| = \mathcal{O}(\infty)$, while $|\partial_t V|, |\partial_x V| = O(\epsilon^{-1})$ as $\epsilon \to 0$, so that $|\partial_x \tilde{V}| \ll |\partial_x V|$ and $|\mathcal{B}| \ll |\partial_t V|$. We now look for solutions of (1) in the form

$$\tilde{V}(x,\xi) = V^{(0)} + \epsilon V^{(1)}(x,\xi) + \epsilon^2 V^{(2)}(x,\xi) + O(\epsilon^2), \tag{40}$$

$$\tilde{T}(x,\xi) = T^{(0)} + \epsilon T^{(1)}(x,\xi) + \epsilon^2 T^{(2)}(x,\xi) + O(\epsilon^2), \tag{41}$$

subject to the following boundary conditions at $x = x_0$:

$$p(x_0, \xi) = p_0 + \epsilon \tilde{\Pi}(\xi), \quad \tilde{T}(x_0, \xi) = \epsilon \xi, \tag{42}$$

where $V^{(0)}$ refers to the uniform reference state and $\tilde{\Pi}(\xi) = O(1)$. Introducing the expansions (40) and (41) into (4), (5), (8) and (42), using the current notations, and equating to zero the coefficients of various powers of ϵ, we arrive at the following $O(1)$, $O(\epsilon)$ and $O(\epsilon^2)$ problems,

$O(1)$ problem:
$$\partial_x T^{(0)} = 1/a_0, \quad T^{(0)}(x_0, \xi) = 0; \tag{43}$$

$O(\epsilon)$ problem:
$$\left.\begin{array}{c} \partial_\xi \rho^{(1)} = (\rho_0/a_0)\partial_\xi u^{(1)} = (1/a_0^2)(\partial_\xi p^{(1)}), \quad \partial_\xi \sigma^{(1)} = 0, \\ \partial_x T^{(1)} = -(u^{(1)} + a^{(1)})/a_0^2, \\ \partial_x p^{(1)} + \rho_0 a_0 (\partial_x u^{(1)}) + \Omega a_0 u^{(1)} + (\gamma - 1)\Lambda^{(1)}/a_0 = 0, \\ p^{(1)}(x_0, \xi) = \tilde{\Pi}, \quad T^{(1)}(x_0, \xi) = \xi; \end{array}\right\} \tag{44}$$

$O(\epsilon^2)$ problem:

$$\left.\begin{array}{rl} \partial_\xi(a_0\rho^{(2)} - \rho_0 u^{(2)}) + a^{(1)}(\partial_\xi \rho^{(1)}) - \rho^{(1)}(\partial_\xi u^{(1)}) & \\ +\rho_0 a_0(\partial_x u^{(1)} + \Omega u^{(1)})(\partial_\xi T^{(1)}) & = 0, \\ \rho_0 \partial_\xi(\rho_0 a_0 u^{(2)} - p^{(2)}) + \rho^{(1)}(\partial_\xi p^{(1)}) + \rho_0^2 a^{(1)}(\partial_\xi u^{(1)}) & \\ +\rho_0 a_0(\partial_x p^{(1)})(\partial_\xi T^{(1)}) & = 0, \\ a_0(\partial_\xi \sigma^{(2)}) + a^{(1)}(\partial_\xi \sigma^{(1)}) - a_0 \Lambda^{(1)}(\partial_\xi T^{(1)}) = 0, & \\ a_0 \partial_\xi(p^{(2)} - \rho_0 a_0 u^{(2)}) + a^{(1)}(\partial_\xi p^{(1)}) - \gamma p^{(1)}(\partial_\xi u^{(1)}) & \\ +\rho_0 a_0\{a_0^2(\partial_x u^{(1)} + \Omega u^{(1)}) + (\gamma - 1)\Lambda^{(1)}\}(\partial_\xi T^{(1)}) & = 0, \\ \partial_x T^{(2)} = a_0^{-3}(u^{(1)} + a^{(1)})^2 - a_0^{-2}(u^{(2)} + a^{(2)}), & \\ \partial_x p^{(2)} + \rho_0 a_0(\partial_x u^{(2)}) + (\rho_0 a^{(1)} + a_0 \rho^{(1)})(\partial_x u^{(1)}) & \\ +\Omega \rho_0(a_0 u^{(2)} - a^{(1)} u^{(1)}) + \Omega((\gamma/a_0)p^{(1)} - \rho_0 u^{(1)})u^{(1)} & \\ +(\gamma - 1)\rho_0 \Sigma^{(2)}/a_0 - (\gamma - 1)\rho_0 \Lambda^{(1)} a^{(1)}/a_0^2 & = 0, \\ p^{(2)}(x_0, \xi) = 0, \quad T^{(2)}(x_0, \xi) & = 0, \end{array}\right\} \tag{45}$$

where

$$\left.\begin{array}{rl} a^{(1)} & = (\gamma p^{(1)} - a_0^2 \rho^{(1)})/2\rho_0 a_0, \\ a^{(2)} & = (2\rho_0 a_0)^{-1}(\gamma p^{(2)} - a_0^2 \rho^{(2)}) \\ & \quad -(8\rho_0^2 a_0^3)^{-1}\{\gamma^2 p^{(1)^2} - 3a_0^4 \rho^{(1)^2} + 2\gamma a_0^2 p^{(1)} \rho^{(1)}\}, \\ \Lambda^{(1)} & = (\partial_p Q)_0 p^{(1)} + (\partial_\rho Q)_0 \rho^{(1)} + (\partial_\sigma Q)_0 \sigma^{(1)}, \\ \Sigma^{(2)} & = \frac{1}{2}((\partial^2_{pp} Q)_0 p^{(1)^2} + (\partial^2_{\rho\rho} Q)_0 \rho^{(1)^2} + (\partial^2_{\sigma\sigma} Q)_0 \sigma^{(1)^2}) \\ & \quad +(\partial^2_{p\rho} Q)_0 p^{(1)} \rho^{(1)} + (\partial^2_{p\sigma} Q)_0 p^{(1)} \sigma^{(1)} + (\partial^2_{\rho\sigma} Q)_0 \rho^{(1)} \sigma^{(1)} \\ & \quad +(\partial_p Q)_0 p^{(2)} + (\partial_\rho Q)_0 \rho^{(2)} + (\partial_\sigma Q)_0 \sigma^{(2)}. \end{array}\right\} \tag{46}$$

(a) *First-order solution*

In view of the condition that the front $\xi = 0$ is moving into a uniform state at rest, equations (43), $(44)_{1,2}$ imply

$$T^{(0)} = (x - x_0)/a_0, \quad u^{(1)} = p^{(1)}/\rho_0 a_0, \quad \rho^{(1)} = p^{(1)}/a_0^2, \quad \sigma^{(1)} = 0. \tag{47}$$

On using (47) and (2) in $(44)_4$ we obtain the following transport equation for $p^{(1)}$

$$\partial_x p^{(1)} + (\alpha + \Omega/2)p^{(1)} = 0, \tag{48}$$

where α is the same as in (14). Equation (48) yields on integration, subject to $(44)_5$, that

$$p^{(1)} = \tilde{\Pi}(\xi)\psi(x), \tag{49}$$

where $\psi(x)$ is the same as in (16). Equation $(44)_3$, in view of $(44)_6$, (47) and (49), is integrated to yield

$$T^{(1)} = \xi - (\gamma+1)(2\rho_0 a_0^3)^{-1}\tilde{\Pi}(\xi)J, \tag{50}$$

where J is the same as in (17). However, the lowest-order asymptotic solutions for p and T, correct up to $O(\epsilon)$, are similar to the solutions obtained in the § 3 *b* (see equations (16) and (17)), and therefore the discussion in respect of shock formation and its subsequent propagation follows on parallel lines.

(*b*) *Second-order solution*

It is of interest to enquire into the behaviour of the second-order solution. Indeed, equations $(45)_{1-5}$, on using (46),(49), (50), $(45)_8$, and the conditions at ξ=0, yield on integration

$$\left.\begin{aligned}
\rho^{(2)} &= \frac{1}{a_0^2}\left\{p^{(2)} - \frac{(\gamma-1)}{2\rho_0 a_0^2}\tilde{\Pi}^2(\xi)\psi^2(x) - \frac{\alpha(\gamma+1)}{2\rho_0 a_0^2}\tilde{\Pi}^2(\xi)\psi(x)J(x)\right.\\
&\qquad\left. + 2\alpha a_0 P(\xi)\psi(x)\right\},\\
u^{(2)} &= \frac{1}{\rho_0 a_0}p^{(2)} - \frac{(\gamma+1)}{4\rho_0^2 a_0^3}\tilde{\Pi}^2\psi^2 - \frac{(\gamma+1)}{4\rho_0^2 a_0^3}(\alpha+\tfrac{1}{2}\Omega)\tilde{\Pi}^2\psi J\\
&\qquad + \tfrac{1}{\rho_0}(\alpha+\tfrac{1}{2}\Omega)P(\xi)\psi,\\
\sigma^{(2)} &= \frac{2\alpha a_0}{\rho_0(\gamma-1)}P\psi - \frac{(\gamma+1)}{(\gamma-1)}\frac{\alpha}{\rho_0^2 a_0^2}\tilde{\Pi}^2\psi J,\\
a^{(2)} &= \frac{1}{8\rho_0^2 a_0^3}\{4(\gamma-1)\rho_0 a_0^2 p^{(2)} - (\gamma^2-1)\tilde{\Pi}^2\psi^2 - 8\rho_0 a_0^3\alpha P\psi + 2(\gamma+1)\alpha\tilde{\Pi}^2\psi J\},\\
T^{(2)} &= -\frac{\gamma+1}{2\rho_0 a_0^3}\int_{x_0}^{x} p^{(2)}(s,\xi)\,\mathrm{d}s + \frac{(\gamma+1)\tilde{\Pi}^2}{8\rho_0^2 a_0^5}\left\{3(\gamma+1)\int_{x_0}^{x}\psi^2(s)\,\mathrm{d}s\right.\\
&\qquad\left. + m\int_{x_0}^{x}\frac{\psi(s)J(s)}{s}\,\mathrm{d}s - \frac{mP}{2\rho_0 a_0^2}\int_{x_0}^{x}\frac{\psi(s)}{s}\,\mathrm{d}s\right\},
\end{aligned}\right\} \tag{51}$$

where
$$P(\xi) = \int_0^{\xi}\tilde{\Pi}(s)\,\mathrm{d}s.$$

Equations (51) show that the second-order flow entities are known if we can determine $p^{(2)}$. This is supplied by $(45)_6$, which on using (51) simplifies to the following transport equation

$$\partial_x p^{(2)} + (\alpha + \frac{m}{2x})p^{(2)} + \left\{(\gamma-7)\frac{m}{2x} - (5\gamma+1)\alpha\right\}\frac{\tilde{\Pi}^2\psi^2}{8\rho_0 a_0^2}$$
$$-\left\{\frac{m(2-m)a_0}{8x^2} + \frac{(\gamma+3)}{(\gamma-1)}\frac{a_0\alpha^2}{2} + \frac{\alpha}{\tau}\right\}\left\{P(\xi)\psi(x) - \frac{\gamma+1}{4\rho_0 a_0^3}\tilde{\Pi}^2\psi J\right\} = 0.$$

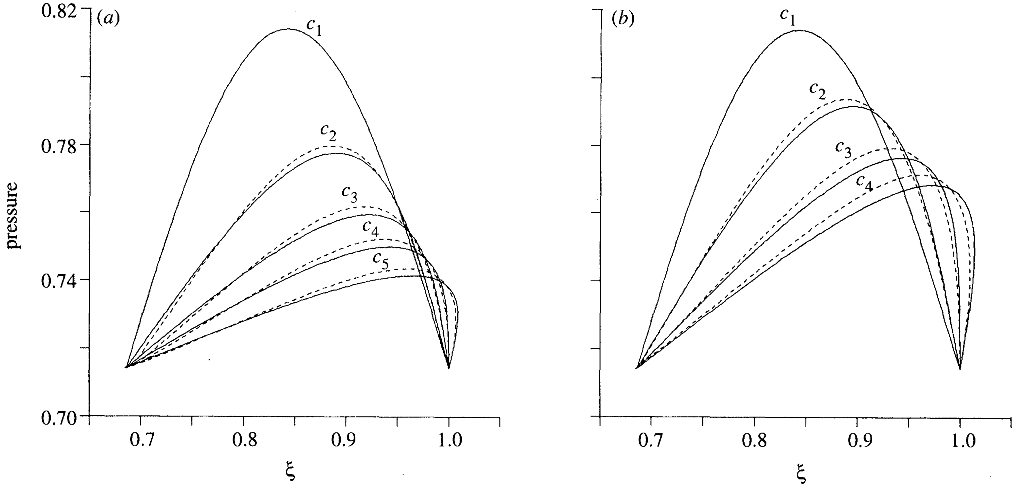


Figure 6. (a) Solution for the pressure $\hat{p}$ in the pulse region of a spherically symmetric ($m = 2$) flow of a relaxing gas at various distances. Pressure distribution, up to first and second order, is represented by the solid and dashed lines respectively; $\epsilon = 0.1$, $\gamma = 1.4$, $\hat{\alpha} = 0.1$. c_1: $\hat{x} = 1.0$; c_2: $\hat{x} = 1.5$; c_3: $\hat{x} = 2.0$; c_4: $\hat{x} = 2.42253$; c_5: $\hat{x} = 3.0$. (b) Solution for the pressure $\hat{p}$ in the pulse region of a cylindrically symmetric ($m = 1$) flow of a relaxing gas at various distances. Pressure distribution, up to first and second order, is represented by the solid and dashed lines respectively; $\epsilon = 0.1$, $\gamma = 1.4$, $\hat{\alpha} = 0.1$. c_1: $\hat{x} = 1.0$; c_2: $\hat{x} = 1.5$; c_3: $\hat{x} = 2.06756$; c_4: $\hat{x} = 2.5$.

This can be integrated subject to the bounndary condition $(45)_7$ to yield

$$
\begin{aligned}
p^{(2)} = \Bigg\{ &\frac{m(m-2)(\gamma+1)}{32} \int_{x_0}^{x} \frac{J(s)}{s^2}\,\mathrm{d}s - \frac{(\gamma+1)\alpha}{8}\left(\frac{\gamma+3}{\gamma-1}\alpha + \frac{2\alpha}{a_0\tau}\right) \int_{x_0}^{x} J(s)\,\mathrm{d}s \\
&-(\gamma-7)\frac{m}{16}\int_{x_0}^{x} \frac{\psi(s)}{s}\,\mathrm{d}s + (5\gamma+1)\frac{\alpha}{8}\int_{x_0}^{x} \psi(s) ds \Bigg\} \frac{\tilde{\Pi}^2 \psi}{\rho_0 a_0^2} \\
&+\frac{1}{2}\left\{\frac{\gamma+3}{\gamma-1} a_0 \alpha^2 - \frac{m(2-m)}{4x}\frac{a_0}{x_0} + \frac{2\alpha}{\tau}\right\}(x - x_0)\psi P(\xi). \qquad (52)
\end{aligned}
$$

In contrast to the first-order solution, equations (51) and (52) show that the second-order solution depends on the integral $P(\xi)$ and consequently on the precursor wavelets; however, the conditions on the leading wavelet remain uninfluenced by the precursor wavelets. For a small-amplitude pulse with $\tilde{\Pi}(\xi) = \rho_0 a_0^2 \sin(\xi a_0/x_0)$, $0 \leqslant a_0\xi/x_0 \leqslant \pi$, the nonlinear distortion of the initial pressure profile, given by $(42)_1$, valid up to the first and second-order approximations, is depicted in figure 6 at various distances x.

5. Resonantly interacting waves

A systematic asymptotic theory for resonantly interacting weakly nonlinear hyperbolic waves, which includes the theory of non-resonantly interacting waves (Hunter & Keller 1983) as a special case, has been proposed by Majda & Rosales (1984) and Hunter *et al.* (1986). This approach has enabled us to analyse situations where many waves coexist and interact with one another resonantly. This section is concerned with the propagation of resonant wave modes produced by a small-amplitude high-frequency boundary perturbation about the uniform state, $V_i = V_i^{(0)}$, of a relaxing gas motion governed by the hyperbolic system (1).

The matrix $\mathcal{A}$ in (1) has eigenvalues $\lambda_{1,2} = u \pm a$, and $\lambda_{3,4} = u$; let $L^{(i)}, R^{(i)}$ be the left and right eigenvectors associated with these eigenvalues. We introduce the phase functions ϕ_J ($J =$ 1,2,3) and the corresponding fast variables, $\theta_J = \phi_J/\epsilon$, associated with the three distinct modes of wave propagation on which the solution vector V depends; here ϵ is the small parameter which is same as defined in the last section. We now seek an asymptotic solution of (1) having the form

$$V_i(x,t) = V_{i0} + \epsilon V_i^{(1)}(x,t,\theta) + \epsilon^2 V_i^{(2)}(x,t,\theta) + O(\epsilon^3), \tag{53}$$

satisfying the boundary datum

$$V_i(x_0,t) = V_{i0} + \epsilon V_{i0}^{(1)}(t,t/\epsilon) + \epsilon^2 V_{i0}^{(2)}(t,t/\epsilon) + O(\epsilon^3), \tag{54}$$

where $V_{i0}^{(1)}$ and $V_{i0}^{(2)}$ are arbitrary, continuously differentiable and bounded functions, and $\theta = (\theta_1, \theta_2, \theta_3)$. The dependence of $V_i^{(1)}$ and $V_i^{(2)}$ on fast variables θ_j allows us to deal with the interactions among various wave modes. Here, we shall use the method of multiple scales, in which x, t and θ are treated as independent variables. Thus, if $V^{(1)}$ is to give the leading-order asymptotic behaviour of V uniformly in x and t, $V^{(1)}$ must satisfy appropriate conditions. In fact, to this order, a sufficient condition for the asymptotic expansion (53) to be uniformly valid in x, for times of order ϵ^{-1}, is that $V^{(1)}, V^{(2)}$ and their derivatives with respect to x and θ are bounded functions of x and θ as $\epsilon \to 0$, and they are averageable over the fast variables θ_J. It may be recalled that averaging procedures are often used in the derivation of uniformly valid asymptotic solutions to nonlinear differential equations. The basic idea underlying this procedure, which renders the asymptotic expansion uniformly valid, is to seperate the rapidly varying part of the solution from the slowly varying part; this is accomplished by averaging the solution with respect to the rapid variable. Using (53) in (1) and then equating to zero the coefficients of ϵ^0 and ϵ in the resulting expression, we obtain

$$((\mathcal{A}_{ik})_0 \, (\partial_x \phi_J) + \delta_{ik} \, (\partial_t \phi_J))(\partial_{\theta_J} V_k^{(1)}) = 0, \tag{55}$$

$$\begin{aligned}((\mathcal{A}_{ik})_0 \, (\partial_x \phi_J) + \delta_{ik} \, (\partial_t \phi_J))(\partial_{\theta_J} V_k^{(2)}) + \partial_t V_i^{(1)} + (\mathcal{A}_{ik})_0 \, (\partial_x V_k^{(1)}) \\ + (\partial_{V_\ell} \mathcal{A}_{ik})_0 \, V_\ell^{(1)} (\partial_{\theta_J} V_k^{(1)}) \, (\partial_x \phi_J) + (\partial_{V_k} \mathcal{B}_i)_0 V_k^{(1)} = 0,\end{aligned} \tag{56}$$

where the subscript '0' denotes a value evaluated at $V_i = V_{i0}$; the indices i, k and ℓ run between 1 and 4 while the index 'J' runs between 1 and 3. Assuming that each term in the sum over 'J' in (55) vanishes separately, so that $\phi_J =$ const. are the characteristic curves belonging to the family $\mathrm{d}x/\mathrm{d}t = (\lambda_J)_0$; the characteristic families $\phi_1 =$ const. and $\phi_2 =$ const. define respectively the eigenvalues $(\lambda_1)_0$ and $(\lambda_2)_0$ of $\mathcal{A}_0$, each of multiplicity one, while the family $\phi_3 =$ const. defines the eigenvalue, $(\lambda_3)_0 = 0$, of multiplicity two. Thus, the phase functions $\phi_J(x,t)$, satisfying the boundary conditions $\phi_{1,2}(x_0,t) = t$, $\phi_3(x,t) = 0$ are given by $\phi_{1,2}(x,t) = t \mp (x - x_0)/a_0$, $\phi_3(x,t) = (x - x_0)/a_0$. The left and right eigenvectors of

$\mathcal{A}_0$ corresponding to the eigenvalues $\lambda_{1,2} = \pm a_0$ and $\lambda_3 = 0$ are given by

$$\begin{aligned} L^{(1)} &= (0, \rho_0 a_0, 0, 1), & R^{(1)} &= (a_0^{-2}, 1/\rho_0 a_0, 0, 1)^{\mathrm{T}}, \\ L^{(2)} &= (0, -\rho_0 a_0, 0, 1), & R^{(2)} &= (a_0^{-2}, -1/\rho_0 a_0, 0, 1)^{\mathrm{T}}, \\ L^{(3)} &= (a_0^2, 0, 0, -1), & R^{(3)} &= (a_0^{-2}, 0, 0, 0)^{\mathrm{T}}, \\ L^{(4)} &= (0, 0, \rho_0, 0), & R^{(4)} &= (0, 0, 1/\rho_0, 0)^{\mathrm{T}}. \end{aligned}$$

Subsequently $\partial_{\theta_1} V^{(1)}$ and $\partial_{\theta_2} V^{(1)}$ are parallel to $R^{(1)}$ and $R^{(2)}$ respectively, while $\partial_{\theta_3} V^{(1)}$ can be expressed linearly in terms of $R^{(3)}$ and $R^{(4)}$. Indeed, $V^{(1)}$ may be written as

$$V_i^{(1)}(x, t, \theta) = \overline{V}_i(x, t) + \beta_j\, R_i^{(j)}, \tag{57}$$

where $\overline{V}(x, t)$ is an arbitrary vector and β_j, which are functions of x, t and θ_J, are arbitrary scalar functions such that $\beta_J = \beta_J(x, t, \theta_J)$ and $\beta_4 = \beta_4(x, t, \theta_3)$; in fact, β_1 and β_2 are the amplitudes of waves which propagate along the characteristic families $\mathrm{d}x/\mathrm{d}t = a_0$ and $\mathrm{d}x/\mathrm{d}t = -a_0$, respectively, while the amplitudes β_3 and β_4 correspond to the characteristic family $\mathrm{d}x/\mathrm{d}t = 0$.

When $V_{i0}^{(1)}$ are denoted by the continuously differentiable and bounded functions $f_i(t, t/\epsilon)$, evaluation of (57) at $x = x_0$ yields

$$\left.\begin{aligned} \overline{\rho}_0 + a_0^{-2}(\beta_{10} + \beta_{20} + \beta_{30}) = f_1, \quad \overline{u}_0 + (\rho_0 a_0)^{-1}(\beta_{10} - \beta_{20}) = f_2, \\ \overline{\sigma}_0 + \rho_0^{-1}\beta_{40} = f_3, \quad \overline{p}_0 + \beta_{10} + \beta_{20} = f_4, \end{aligned}\right\} \tag{58}$$

where $\overline{V}_{i0} = \overline{V}_i(x_0, t)$ and $\beta_{j0} = \beta_j(x_0, t)$. Without loss of generality, the vector $\overline{V}$ may be chosen to be the mean of $V^{(1)}$ with respect to θ, so that the mean values of β_J and β_4 with respect to θ_J and θ_3, respectively, are zero, i.e.

$$\lim_{T\to\infty}\left\{\frac{1}{T}\int_0^T \beta_J(x, t, \theta_J)\,\mathrm{d}\theta_J\right\} = 0, \quad \lim_{T\to\infty}\left\{\frac{1}{T}\int_0^T \beta_4(x, t, \theta_3)\,\mathrm{d}\theta_3\right\} = 0. \tag{59}$$

Using (57) in (56) and averaging the resulting equation with respect to θ, we obtain the following linear system of equations for $\overline{\rho}$, $\overline{u}$, $\overline{\sigma}$ and $\overline{p}$

$$\left.\begin{aligned} \partial_t\overline{\rho} + \rho_0\partial_x\overline{u} + \Omega\rho_0\overline{u} = 0, \quad \partial_t\overline{u} + \rho_0^{-1}\partial_x\overline{p} = 0, \\ \partial_t\overline{\sigma} = (\partial_\rho Q)_0\overline{\rho} + (\partial_p Q)_0\overline{p} + (\partial_\sigma Q)_0\overline{\sigma}, \\ \partial_t\overline{p} + \rho_0 a_0^2(\partial_x\overline{u} + \Omega\overline{u}) + (\gamma - 1)\rho_0(\partial_t\overline{\sigma}) = 0. \end{aligned}\right\} \tag{60}$$

Boundary conditions for $\overline{V}_i$ at $x = x_0$ can be obtained by evaluating (57) at $x = x_0$, and then taking the average with respect to the fast variable $t^* = t/\epsilon$ subject to the requirement (59), i.e.

$$\overline{V}_{i0} = \lim_{T\to\infty}\left\{\frac{1}{T}\int_0^T f_i(t, t^*)\,\mathrm{d}t^*\right\}.$$

By routine elimination of $\overline{\rho}$, $\overline{u}$, and $\overline{\sigma}$ in (60), we find on using (2) that $\overline{p}$ satisfies the third-order partial differential equation,

$$\begin{aligned} \partial_t(\partial_t - a_0\partial_x)(\partial_t + a_0\partial_x)\overline{p} + \xi(\partial_t - a_0\surd(\chi/\xi)\,\partial_x)(\partial_t + a_0\surd(\chi/\xi)\,\partial_x)\overline{p} \\ -(m a_0^2/x)\partial_x(\partial_t + \chi)\overline{p} = 0, \end{aligned} \tag{61}$$

where $\xi = (2\alpha\gamma a_0\tau + \gamma - 1)/(\gamma - 1)\tau$ and $\chi = (2\alpha a_0\tau + \gamma - 1)/(\gamma - 1)\tau$. We seek a solution of (61) satisfying the initial and boundary conditions,

$$\overline{p}(x,0)=0, \quad \overline{p}_0=\overline{p}(x_0,t), \quad \overline{p}_{0x}=-\rho_0\overline{u}_{0t}, \tag{62}$$

$$\overline{p}_{0xx}=\frac{1}{a_0^2}\left\{\overline{p}_{0tt}-\frac{a_0^2}{\tau}\frac{(\gamma-1+2\alpha a_0\tau)}{(\gamma-1)}\overline{\rho}_{0t}+\frac{\gamma-1+2\alpha\gamma a_0\tau}{(\gamma-1)\tau}\overline{p}_{0t}-\frac{m\gamma p_0}{x}\overline{u}_{0t}\right\},$$

where the subscripts denote partial derivatives with respect to the indicated variables. The differential equation (61) satisfying (62) does not seem to have a solution interms of the well-known functions of mathematical physics; however, it can be solved with the aid of Laplace transform. Let $\tilde{V}_i(x,s)$ be the Laplace transform of the mean quantity $\overline{V}_i(x,t)$, i.e.

$$\tilde{V}_i(x,s)=\int_0^\infty \overline{V}_i(x,t)\mathrm{e}^{-st}\,\mathrm{d}t.$$

Then it follows that the required solution of (61) satisfying the transformed initial and boundary conditions (62),

$$\tilde{p}(x,\infty)=0, \quad \tilde{p}_0=\tilde{p}(x_0,s) \quad \text{and} \quad \tilde{p}_{0x}=\left.\frac{\partial\tilde{p}}{\partial x}\right|_{x=x_0}=-\rho_0 s\tilde{u}(x_0,s), \tag{63}$$

can be expressed as

$$\tilde{p}(x,s)=x^n\mathcal{Z}_n(x\sqrt{\Delta}), \quad n=\tfrac{1}{2}(1-m),$$

where

$$\Delta=(s^2/a_0^2)\{(\gamma-1)(1+s\tau)+2\alpha\gamma a_0\tau\}\{(\gamma-1)(1+s\tau)+2\alpha a_0\tau\}^{-1},$$

and

$$\mathcal{Z}_n(x\sqrt{\Delta})=\begin{cases}c_{11}(s)I_n(x\sqrt{\Delta})+c_{21}(s)I_{-n}(x\sqrt{\Delta}), & \text{if } n \text{ is not an integer},\\ c_{12}(s)I_n(x\sqrt{\Delta})+c_{22}(s)K_n(x\sqrt{\Delta}), & \text{if } n \text{ is an integer},\end{cases}$$

with $I_n(x)$ and $K_n(x)$ being respectively the modified Bessel functions of first and second kind. Here, the arbitrary functions c_{ij} can be determined using the initial and boundary conditions given in (63). Having obtained $\tilde{p}$, it follows from the Laplace inversion theorem that

$$\overline{p}(x,t)=\frac{1}{2\pi\mathrm{i}}\lim_{Y\to\infty}\int_{c_1-\mathrm{i}Y}^{c_1+\mathrm{i}Y}\tilde{p}(x,s)\mathrm{e}^{st}\,\mathrm{d}s, \tag{64}$$

where c_1 is a real constant which is greater than the real part of singularities, if any, in the integrand of equation (64); the corresponding expressions for $\overline{\rho}$, $\overline{u}$ and $\overline{\sigma}$ follow from (60).

Again using (57) in (56) and averaging the resulting equation with respect to θ_i other than θ_k, and taking scalar product of this equation with $L^{(k)}$ (the left eigenvector of $\mathcal{A}_0$ corresponding to the eigenvalue $(\lambda_k)_0$), we get the following transport equations for the quantities β_k $(k=1,2,3,4)$:

$$\left.\begin{aligned}\partial_{\eta_1}\beta_1-(\gamma+1)(2\rho_0a_0^3)^{-1}\beta_1(\partial_{\theta_1}\beta_1)+h_1(\partial_{\theta_1}\beta_1)+D_1\beta_1+T_1=0,\\ \partial_{\eta_2}\beta_2+(\gamma+1)(2\rho_0a_0^3)^{-1}\beta_2(\partial_{\theta_2}\beta_2)+h_2(\partial_{\theta_2}\beta_2)+D_2\beta_2+T_2=0,\\ \partial_t\beta_3+(\overline{u}/a_0)(\partial_{\theta_3}\beta_3)-(\gamma-1)\rho_0a_0^{-2}(\partial_\rho Q)_0\beta_3-(\gamma-1)(\partial_\sigma Q)_0\beta_4=0,\\ \partial_t\beta_4+(\overline{u}/a_0)(\partial_{\theta_3}\beta_4)-\rho_0a_0^{-2}(\partial_\rho Q)_0\beta_3-(\partial_\sigma Q)_0\beta_4=0,\end{aligned}\right\} \tag{65}$$

where

$$\partial_{\eta_1} = \partial_x + a_0^{-1}\partial_t, \quad \partial_{\eta_2} = \partial_x - a_0^{-1}\partial_t,$$

$$h_1 = \overline{\rho}/2\rho_0 a_0 - \gamma\overline{p}/2\rho_0 a_0^3 - \overline{u}/a_0^2, \quad h_2 = -\overline{\rho}/2\rho_0 a_0 + \gamma\overline{p}/2\rho_0 a_0^3 - \overline{u}/a_0^2,$$

$$D_1 = \alpha + \tfrac{1}{2}\Omega, \quad D_2 = \tfrac{1}{2}\Omega - \alpha,$$

$$T_1 = -(\rho_0 a_0^3)^{-1} \lim_{T\to\infty}\left\{\frac{1}{T^2}\int_0^T\int_0^T \beta_3 \frac{\partial\beta_2}{\partial\theta_2}\,\mathrm{d}\theta_2\mathrm{d}\theta_3\right\},$$

$$T_2 = (\rho_0 a_0^3)^{-1} \lim_{T\to\infty}\left\{\frac{1}{T^2}\int_0^T\int_0^T \beta_3 \frac{\partial\beta_1}{\partial\theta_1}\,\mathrm{d}\theta_1\mathrm{d}\theta_3\right\},$$

with α and Ω being the same as in equation (14). The boundary conditions for equations in (65) are recovered from (58), and have the form

$$\left.\begin{aligned} \beta_{10} = \tfrac{1}{2}\{\rho_0 a_0(f_2 - \overline{u}_0) + f_4 - \overline{p}_0\}, \quad \beta_{20} = \tfrac{1}{2}\{f_4 - \overline{p}_0 + \rho_0 a_0(\overline{u}_0 - f_2)\}, \\ \beta_{30} = a_0^2(f_1 - \overline{\rho}_0) - f_4 + \overline{p}_0, \quad \beta_{40} = \rho_0(f_3 - \overline{\sigma}_0). \end{aligned}\right\} \tag{66}$$

Under the assumption that the waves with phases ϕ_2 and ϕ_3 (respectively, ϕ_1 and ϕ_3) are non-interacting resonantly, the integrals involved in the expression for T_1 (respectively, T_2) can be evaluated by treating θ_2 and θ_3 (respectively, θ_1 and θ_3) as linearly independent; indeed, both T_1 and T_2 turn out to be zero, and the resulting transport equations $(65)_{1,2}$ are the uncoupled inviscid Burger's equation which are exactly the same as those obtained for acoustical wave mode $\mathrm{d}x/\mathrm{d}t = a_0$ and $\mathrm{d}x/\mathrm{d}t = -a_0$, respectively. Besides this, the integrals T_1 and T_2 are again zero, whenever the functions depend either linearly on θ_J or they are functions of x and t alone. In view of the fact that θ_1, θ_2, θ_3 form a linearly dependent set, i.e. $\theta_3 = \frac{1}{2}(\theta_2 - \theta_1)$, it follows that the coupling terms T_1 and T_2 do not necessarily vanish and, infact, we have

$$\left.\begin{aligned} T_1 &= -\frac{1}{2\rho_0 a_0^3}\lim_{T\to\infty}\left\{\frac{1}{T}\int_0^T \beta_2 \frac{\partial\beta_3}{\partial\theta_3}\bigg|_{\theta_3=(\theta_2-\theta_1)/2}\mathrm{d}\theta_2\right\}, \\ T_2 &= -\frac{1}{2\rho_0 a_0^3}\lim_{T\to\infty}\left\{\frac{1}{T}\int_0^T \beta_1 \frac{\partial\beta_3}{\partial\theta_3}\bigg|_{\theta_3=(\theta_2-\theta_1)/2}\mathrm{d}\theta_1\right\}. \end{aligned}\right\} \tag{67}$$

The entity T_1 in $(65)_1$, which is given by $(67)_1$, represents a contribution to the wave amplitude β_1 on account of nonlinear interaction between waves with phases ϕ_2 and ϕ_3. Similarly, the entity T_2 in $(65)_2$, which is given by $(67)_2$, represents coupling between waves with phases ϕ_1 and ϕ_3; this coupling produces, through nonlinear interaction, a wave with phase ϕ_2. The nonlinear terms proportional to $\beta_1(\partial_{\theta_1}\beta_1)$ and $\beta_2(\partial_{\theta_2}\beta_2)$ in equations $(65)_{1,2}$ account for the nonlinear self-interactions which generate higher harmonics leading to the distortion of the wave profile and consequent shock formation. The absence of nonlinear self-interaction terms in equations $(65)_{3,4}$, which decouple from the acoustical wave components β_1 and β_2, show that the wave amplitudes β_3 and β_4 corresponding to the repeated mode θ_3 are linearly degenerate. Thus, we can solve the linear system $(65)_{3,4}$ for β_3 and β_4 and then use the expression β_3 in the equations for β_1 and β_2.

If in the boundary data (54), all the components are periodic having the same period, then non-resonance always occurs, and the nonlinear equations $(65)_1$ and $(65)_2$ decouple into inviscid Burger's equations, which can be integrated using the boundary conditions $(66)_{1,2}$ to yield

$$\beta_1 = \beta_{10}(\zeta_1)(A/A_0)^{-1/2}\exp(-\hat{\alpha}(\hat{x}-1)), \tag{68}$$

$$\beta_2 = \beta_{20}(\zeta_2)(A/A_0)^{-1/2}\exp(\hat{\alpha}(\hat{x}-1)), \tag{69}$$

where $\hat{\alpha} = \alpha x_0$, $\hat{x} = x/x_0$ and

$$\zeta_1 = \theta_1 + \frac{\gamma+1}{2\rho_0 a_0^3}\int_{x_0}^{x} \beta_{10}(\zeta_1)(A(x')/A_0)^{-1/2}\exp(-\alpha(x'-x_0))\,\mathrm{d}x' - \int_{x_0}^{x} h_1(x',t)\,\mathrm{d}x',$$

$$\zeta_2 = \theta_2 - \frac{\gamma+1}{2\rho_0 a_0^3}\int_{x_0}^{x} \beta_{20}(\zeta_2)(A(x')/A_0)^{-1/2}\exp(\alpha(x'-x_0))\,\mathrm{d}x' - \int_{x_0}^{x} h_2(x',t)\,\mathrm{d}x'.$$

It may be noted that by setting $\overline{V} = 0, \zeta = \xi$ and $\beta_{10} = \tilde{\Pi}$, equation (68) turns out to be the same as the solution (49), which was derived either by using the theory of relatively undistorted waves or the theory of weakly nonlinear geometrical acoustics for a single wave mode. When many waves coexist, the uniformly valid solution to $O(\epsilon)$, satisfying the boundary data (54), is given by (57), (68) and (69).

Let us consider a sinusoidal source at the boundary $\hat{x} = 1$ defined as $V_{\hat{x}=1} = V_0 + \epsilon\rho_0 a_0^2 \sin(\hat{t})R^{(1)}$ with $\hat{t} = a_0 t/\epsilon x_0$. In view of the foregoing discussion, it follows immediately that when all the wave modes are excited by this source, the resultant waveform of the pressure profile to $O(\epsilon)$ is given by

$$p = p_0 + \epsilon(\beta_1 + \beta_2), \tag{70}$$

where β_1 and β_2 are given by (68) and (69) with $\beta_{10} = \beta_{20} = \rho_0 a_0^2 \sin(\hat{t})$. It may be noted that the solution (70) holds for distances $\hat{x} < \hat{x}_s = \min(\hat{x}_{s1}, \hat{x}_{s2})$, where $\hat{x}_{s1}$ and $\hat{x}_{s2}$ are the distances at which the solutions (68) and (69) develop shocks. For the spherical case, the variation of the pressure profile, given by (70), over a complete cycle is exhibited in figure 7 at distances before the formation of a shock wave. The waveforms at $\hat{x} > 1$ show nonlinear effects. In fact, as the wave propagates, it undergoes distortion, the amplitude of the crest being greater than that of the trough. It is observed that the compression phase of the wave profile, with a rounded crest peak, steepens at its ends to yield shocks, while the rarefaction phase, which follows the compression phase, flattens. Further, in the absence of relaxation ($\hat{\alpha} = 0$), the compressive and expansive phases of the distorted waveform exhibit symmetry, which, however, gradually disappears when the relaxation effects are introduced (i.e. $\hat{\alpha} \neq 0$).

6. Far field behaviour

Since at large times, far from the piston location, any nonlinear convection is associated with the low-frequency characteristics, we shall study in this section the asymptotic wave motion described by the system (1) in the low-frequency domain. Let ω be a small parameter defined as $\omega = \tau/\tau_a = L/L_a$, where $L_a = a_0\tau_a$ and $L = a_0\tau$ represent respectively the attenuation length and the characteristic length scales for the medium. When $\omega^2 \ll 1$, which means that the time and distances considered are large in comparison to the relaxation time and the relaxation length, the situation corresponds to a low-frequency wave process (Fusco 1982). As the principal signal (in this region) is centred on the equilibrium characteristic, the system (1) is

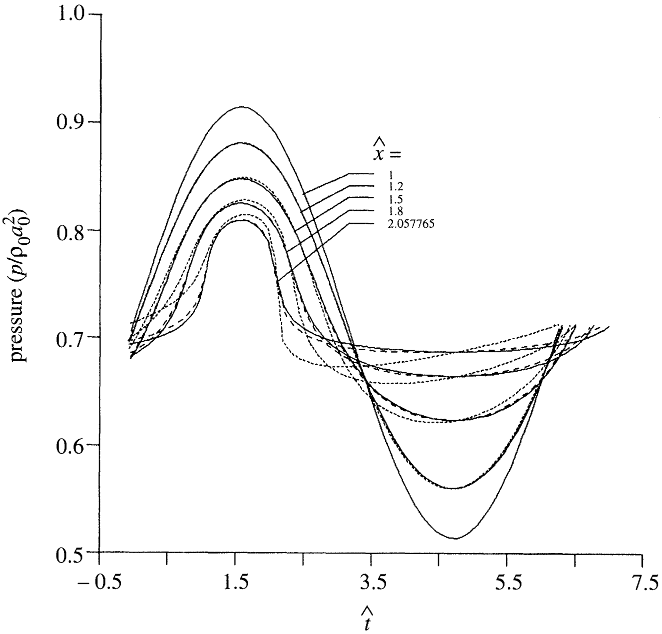


Figure 7. Development of the resultant pressure profile (given by equation (70)) in a spherically symmetric flow ($m = 2$) at distances before the formation of a shock wave when all waves are excited by a sinusoidal source at the boundary $\hat{x} = 1$; $\epsilon = 0.1$. Solid curves refer to a non-relaxing gas ($\hat{\alpha} = 0$), while the dashed and dotted curves refer to a relaxing gas with ($\hat{\alpha} = 0.05$) and ($\hat{\alpha} = 0.3$) respectively. ——, $\hat{\alpha} = 0.0$; – – – –, $\hat{\alpha} = 0.05$; - - - - - - -, $\hat{\alpha} = 0.3$.

approximated by the following reduced system:

$$\left.\begin{array}{c}\partial_t\rho + u\partial_x\rho + \rho\partial_x u + \Omega\rho u = 0, \quad \partial_t u + u\partial_x u + \rho^{-1}\partial_x p = 0,\\ \partial_t p + u\partial_x p + \gamma p(\partial_x u + \Omega u) + (\gamma - 1)\rho(\partial_t\sigma + u\partial_x\sigma) = 0, \quad Q(p, p, \sigma) = 0.\end{array}\right\} \tag{71}$$

In order to study the influence of non-equilibrium relaxation in (1) on the wave motion, associated with (71), we consider the following stretching of the independent variables

$$\tilde{x} = \omega^2 x, \quad \tilde{t} = \omega^2 t.$$

When expressed in terms of $\tilde{x}$ and $\tilde{t}$, and then suppressing the tilde sign, the system (1) yields the following set of equations

$$\left.\begin{array}{c}\partial_t\rho + u\partial_x\rho + \rho\partial_x u + \Omega\rho u = 0, \quad \partial_t u + u\partial_x u + \rho^{-1}\partial_x p = 0,\\ \partial_t p + u\partial_x p + \gamma p(\partial_x u + \Omega u) + (\gamma - 1)\rho(\partial_t\sigma + u\partial_x\sigma) = 0,\\ \omega^2(\partial_t\sigma + u\partial_x\sigma) = Q(p, \rho, \sigma).\end{array}\right\} \tag{72}$$

In the limit $\omega \to 0$, the above system yields the reduced system (71), and, therefore, the transformed system (72) may be regarded as a perturbed problem of an equilibrium state characterized by (71).

We now look for an asymptotic solution of (72) exhibiting the character of a progressive wave, i.e.

$$f(x, t) = f_0 + \omega f^{(1)}(x, t, \xi) + \omega^2 f^{(2)}(x, t, \xi) + \dots, \tag{73}$$

where f may denote any of the dependent variables ρ, u, σ and p; f_0 refers to the known constant state, $\xi = \phi(x,t)/\omega$ is a fast variable, and $\phi(x,t)$ is the phase function to be determined.

By introducing (73) and the Taylor's expansion of Q about the uniform state $(\rho_0, 0, \sigma_0, p_0)$ into the transformed system (72) and equating to zero the coefficients of ω^0, ω^1 and ω^2, we obtain a set of first-order partial differential equations for the first and second-order variables. The system of equations for the first-order variables yields, on solving, the following relations:

$$\left.\begin{aligned} \rho^{(1)} &= \Gamma^{-2} p^{(1)}, \quad u^{(1)} = (\rho_0 \Gamma)^{-1} p^{(1)}, \\ \sigma^{(1)} &= (\gamma - 1)c\{\rho_0(\gamma + (\gamma - 1)c)\}^{-1} p^{(1)}, \end{aligned}\right\} \tag{74}$$

with $\phi(x,t) = t - (x - x_0)/\Gamma$ and $\Gamma = a_0\{(c\gamma + \gamma - c)/(c\gamma^2 - c\gamma + \gamma)\}^{1/2}$. It may be noted that Γ is a characteristic speed related to the reduced system (71). The system of equations for the second-order variables, on multiplying by the left eigenvector corresponding to the eigenvalue Γ and taking into account the relations (74) yields the following transport equation for $p^{(1)}$ in the moving set of coordinates X and ξ,

$$\partial_X p^{(1)} - \lambda p^{(1)} \partial_\xi p^{(1)} + m p^{(1)}/2X = \nu \partial^2_{\xi\xi} p^{(1)}, \tag{75}$$

where $\partial_X = \partial_x + \Gamma^{-1}\partial_t$ and the nonlinear and dissipation coefficients λ and ν are given by $\lambda = (\gamma/\rho_0 \Gamma \Lambda_1)\{\gamma + 1 + 2c(\gamma - 1)\}$ and $\nu = 2c\tau\gamma\Gamma a_0^2\{(\gamma - 1)/\Lambda_1\}^2$, with $\Lambda_1 = 2a_0^2(c\gamma + \gamma - c)$.

Equation (75) is known as the generalized Burger's equation which allows us to study in detail various effects that appear in the propagation of plane ($m = 0$), cylindrical ($m = 1$) and spherical ($m = 2$) waves in a dissipative medium with a quadratic nonlinearity. The reader is referred to Crighton & Scott (1979) for a detailed discussion of the solution of such an equation using the method of matched asymptotic expansions.

Here, we shall consider the situation when the disturbance given at the input is in the form of a harmonic wave

$$p^{(1)}(X_0, \xi) = \Pi \sin(\hat{\xi}), \tag{76}$$

where $\hat{\xi} = \xi a_0/x_0$ and $\Pi = O(1)$. By the substitution, $p^{(1)} = P(X/X_0)^{-m/2}$, equation (75) and the boundary condition (76) are brought to the following form

$$\partial_X P - \lambda (X/X_0)^{-m/2} P \partial_\xi P = \nu \partial^2_{\xi\xi} P, \quad P(X_0, \xi) = \Pi \sin \hat{\xi}. \tag{77}$$

The exact solution of (77), for $m = 0$, is readily obtained by use of the Cole–Hopf transformation, which relates solution of Burger's equation to the solution of linear heat equation. During the first stage of propagation of a sound wave, the dissipative processes do not play important roles, and therefore we can neglect the right side of (77), and the solution of the reduced equation with the stated boundary condition can be written in the form

$$P = \Pi \sin(\hat{\xi} + \Theta P/\Pi), \tag{78}$$

where

$$\Theta = \begin{cases} \{(X/X_0)^{(1/2)} - 1\}/\{(X_s/X_0)^{1/2} - 1\}, & \text{for } m = 1, \\ \log(X/X_0)/\log(\tilde{X}_s/X_0), & \text{for } m = 2, \end{cases} \tag{79}$$

with X_s and $\tilde{X}_s$ being respectively the shock formation distances in a cylindrically or spherically symmetric flow; comparing the value of Θ for a plane wave with the corresponding values for cylindrical and spherical waves, we find that the accumulation of nonlinear effects in cylindrical and spherical waves takes place over large distances on account of geometrical divergence, and a shock is always formed after a finite running length as the dissipation of energy has been neglected.

For cylindrical and spherical cases, equation (77) does not have exact solutions, and therefore further approximations are necessary. Following Webster & Blackstock (1978), we use the method of successive approximations. As a first approximation, the nonlinear term in (77) is neglected; the reduced linear equation with the boundary condition has the solution,

$$P = \Pi \mathrm{e}^{-(\nu_*)(X-X_0)} \sin \hat{\xi}, \tag{80}$$

where $\nu_* = \nu a_0^2/X_0^2$. Substituting the first approximation for P from (80) into (77) and then solving the resulting equation, we get a second approximation for P as

$$P = \Pi \mathrm{e}^{-(\nu_*)(X-X_0)} \sin \hat{\xi} + \tfrac{1}{2}\lambda \Pi^2 a_0 (\nu_* X_0/2)^{(m-2)/2} \Lambda \mathrm{e}^{-(2\nu_*)(2X-X_0)} \sin(2\hat{\xi}), \tag{81}$$

with

$$\Lambda = \begin{cases} \displaystyle\int_{\sqrt{(2\nu_* X_0)}}^{\sqrt{(2\nu_* X)}} \mathrm{e}^{t^2}\,\mathrm{d}t, & \text{if} \quad m = 1, \\ \displaystyle\int_{2\nu_* X_0}^{2\nu_* X} \frac{\mathrm{e}^t}{t}\,\mathrm{d}t, & \text{if} \quad m = 2. \end{cases}$$

Continuing in this way, higher approximations can be made; the approximations beyond the second are indeed very complicated.

In view of equations (78) and (80), Carry (1971) suggested the following approximate form of the solution of (77)

$$P = \Pi \mathrm{e}^{-(\nu_*)(X-X_0)} \sin(\hat{\xi} + \Theta P/\Pi), \tag{82}$$

where Θ is the same as given in (79). This is an approximate solution of equation (77) provided $\Theta \exp(-\nu_*(X - X_0)) \ll 1$. The explicit solution for P obtained from (82) can be written in the form of a Fourier series

$$P = \Pi \mathrm{e}^{-(\nu_*)(X-X_0)} \sum_{n=1}^{\infty} \frac{2}{n\Theta_*} J_n(n\Theta_*) \sin(n\hat{\xi}), \tag{83}$$

where $\Theta_* = \Theta \mathrm{e}^{-(\nu_*)(X-X_0)}$ and J_n is the Bessel function of nth order. This gives exactly the same result for the first and second harmonics as obtained by successive approximations, provided $(\nu_*)(X - X_0) \ll 1$, namely

$$P = \Pi \mathrm{e}^{-(\nu_*)(X-X_0)} \sin(\hat{\xi}) + \tfrac{1}{2}\Pi \Theta a_0 X^{(m-2)/2} \mathrm{e}^{-2\nu_*(X-X_0)} \sin(2\hat{\xi}),$$

which describes the attenuation of harmonics, the amplitudes of which change approximately according to the rule $\exp(-n\nu_*(X - X_0))$, where n refers to the number of harmonics.

7. Conclusions

This article uses the relatively undistorted wave approximation, the method of nonlinear geometrical acoustics and some related procedures to analyse wave motion influenced by the effects of nonlinear convection, non-equilibrium relaxation, attenuation, dispersion and geometric spreading. An attempt is made to relate and unify these methods, which appear to be quite disjoint, by drawing the connection between the results obtained by using them.

The method of relatively undistorted waves, which makes no assumption on the wave amplitude, is used to obtain a solution of (1) in a region, where the motion associated with an eigenmode is perturbed at the boundary by an applied pressure. To the first-order approximation, the solution in the wave region shows that for each wavelet, the amplitude dispersion and shock formation depend on the amplitude carried by the wavelet; this is in contrast to the behaviour exhibited by the first-order solution obtained by using the method of nonlinear geometrical acoustics, which is limited to small-amplitude high-frequency disturbances. However, in the small-amplitude limit, the relatively undistorted approximation, to the first order, yields solution which agrees with the first-order solution obtained by using the method of nonlinear geometrical acoustics. In this connection, it is worthwhile to mention the fundamental papers of Clarke (1978, 1979), who analyses amplyfying effects of ambient explosion reaction on finite-amplitude waves following a systematic perturbation procedure.

In the small-amplitude limit, conditions within the wave region, which lead to a shock or no shock, depend strongly on the attenuation effects of the relaxation process; the corresponding results on the leading front $\phi = 0$, where (20) is exact, are in agreement with those obtained by Johannesen & Scott (1978). When the shock forms and propagate into an undisturbed region, its location in the weak shock limit is found using Whitham's equal area rule; asymptotic results for the decay of plane, cylindrical and spherical shocks are obtained, which when specialized to non-relaxing gases, agree fully with the earlier results (Whitham 1974). In order to trace the early history of shock decay after its formation, we consider two specific examples in which the small amplitude disturbance at the boundary is either a pulse or a periodic wave.

The distortion of the pulse, as it propagates, is described by equations (25)–(28); the profiles are computed for cylindrical and spherical motions to elucidate the effect of the attenuation rate α, and the results for both relaxing and non-relaxing gasdynamic configurations are shown in figures 1 and 2. A visible depression and flattening of the peak, and an increase in the shock formation distance indicate that the disturbance is undergoing a general attenuation owing to an increase in the relaxation rate or the wavefront curvature. The shock strength after its formation on $\phi = 0$ at $x = x_s$ grows to a maximum strength and then decays in accordance with the asymptotic results (32) for plane, cylindrical and spherical motions. The decay behaviour of cylindrical and spherical shocks is in qualitative agreement with that presented by Clarke (1984) for plane waves. Indeed, there is a general lowering of the shock strength owing to an increase in the relaxation rate or the wavefront curvature. The development of the periodic wave form and the subsequent shock formation, which are described by equations (29)–(32), are exhibited in figures 3a, b. Evolutionary behaviour of the pressure profile before and after the shock formation, depicted in figure 3a, follows a slightly different pattern from that illustrated in figure 3b in the sense that the profile, which eventually folds into itself, develops concavities with its peak slightly advanced; the effect of the attenuation rate α upon the pressure profile is marked, the higher value leading to an enhancement of the distortion of profiles from the initial shape. Variations of the shock strength with distance x for the initial profiles (25) and (29) are exhibited in figure 4. The shock grows to a maximum strength at $x = x_*$; in fact, at any station x, the shock evolving from (29) is stronger than that from the pulse (25). The effect of relaxation rate and the wavefront curvature is to depress the peaks and shift x_* to a larger value; the reduction in shock strength at any station increases with an increase in α. However, in the absence of relaxation, a shock evolving from the periodic waveform decays faster than that from the pulse, as is evident from the asymptotic results (23) and (33). The subtleties concealed by the solution, in § 3 *a*, obtained by using the relatively undistorted approximation to the first order have not however been exhaustively revealed by the studies in § 3 *b*, which analyses the wave motion on the assumption that the disturbances are of small amplitude. We have therefore examined certain aspects in more detail in § 3 *c*. It may be noted that the method of relatively undistorted waves, in § 3 *b*, implies a restriction on the solution in the form of condition (21), which indeed corresponds to the slow modulation approximation. However, we have been unable to predict how this approximation can be used for a finite-amplitude situation to obtain higher-order corrections using a systematic expansion procedure. Nevertheless, in a not so small disturbance situation, some of the essential features of wave motions that finally develop can still be identified by extending the analysis of the preceding subsection to the next order (see § 3 *c*); the solution exhibits that in contrast to the small-amplitude situation both the rate at which amplitude varies on any wavelet and the time taken to form a shock are influenced by the signal carried by the wavelet. Indeed, the shock arrival time on a wavelet increases as compared to the corresponding small-amplitude case; the computed results are shown in figures 5*a*, *b*.

In § 4, we use the theory of nonlinear geometrical acoustics to obtain small-amplitude high-frequency wave like solution of (1) in the form of regular asymptotic expansions. Here we calculate the asymptotic solution up to the second order. The solution correct up to the order $O(\epsilon)$, which describes the effects of amplitude dispersion, pulse distortion and shock formation, is equivalent to the solution obtained in § 3 *b* (see equations (16) and (17)) by using the relatively undistorted approximation to the first order. The second-order correction, in contrast to the $O(\epsilon)$ approximation depends on the precursor wavelets, displaying the phenomenon of dispersion. Numerical results are given showing how the distortion of a sinusoidal pressure profile is influenced by the second-order correction term; the results are depicted in figure 6. Section 5 uses the theory of weakly nonlinear geometrical acoustics to analyse situation where many waves coexist and interact with one another resonantly; indeed, we look for high-frequency wave solutions which are modulated by a slowly varying carrier. The basic idea underlying the procedure, which renders the asymptotic expansion uniformly valid, is to separate the rapidly varying part of the solution from the slowly varying part; this is accomplished by averaging the solution with respect to the fast variables. Transport equations for the amplitudes β_i of waves, which propagate along the characteristic families of the system (1), are in general integro-differential equations; these equations account for possible nonlinear resonant interactions between various wave modes. However, if the interactions among waves are non-resonant, the transport equations are uncoupled inviscid Burger's equations. The coupling terms in the transport equations represent the amount of a gasdynamic wave mode produced through the interaction of another gasdynamic wave and an entropy wave. We notice that in the flow configuration under consideration, a gasdynamic wave interacts only in the presence of an entropy wave. The nonlinear terms in the transport equations account for self interaction leading to the distortion of wave profile and consequent shock formation; the absence of these terms in the transport equations for β_3 and β_4 shows that the corresponding waves are linearly degenerate. The solution obtained in § 3 *a* for a single wave mode is recovered as a special case.

As an illustration, we have discussed the resultant waveform when all wave modes are excited by a sinusoidal source at the boundary. The solution for the resultant pressure profile, which is uniformly valid to $O(\epsilon)$, is computed at distances before the formation of a shock wave and the results are exhibited in figure 7. It is observed that the compression phase of the wave profile, with a rounded crest peak, steepens at its ends to yield shocks, while the rarefaction phase, which follows the compression phase, flattens. Further, in the absence of relaxation ($\alpha = 0$), the compressive and expansive phases of the waveform exhibit symmetry, which, however, gradually disappears when the relaxation effects are introduced.

The work of § 6, which exploits the ideas of previous sections on high-frequency wave process, deals with an asymptotic approach to the low-frequency problem. Some aspects of this problem for a plane motion have been touched upon before; the papers best known are those by Blythe (1969), Ockendon & Spence (1969) and Crighton & Scott (1979). Here, we use a different approach to obtain the evolution equation in a unified manner for plane, cylindrical and spherical motion of a relaxing gas. An approximation through a stretching of the independent variables describes a singular perturbation of the equilibrium state described by the reduced system, which represents a wave motion with propagation speed less than the frozen sound speed; the wave amplitude along characteristic rays of the reduced system satisfies a transport equation of Burger's type, which has been treated using the method of successive approximations. The approximate solution exhibits the property that the nonlinear steepening is diffused by the dissipative mechanism on account of relaxation; the geometric spreading further contributes to this mechanism against any convective steepening. Thus, in contrast to the high-frequency solution in § 3 *a*, the low-frequency solution, to the first order of approximation, remains shock free.

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